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Remarks on Hypo-Elasticity

C. Truesdell

(May 22, 1963)

The difference between elasticity and hypo-elasticity is illustrated by calculating explicitly the acoustic tensor for principal waves in an arbitrary hypo-elastic material. It is shown that all principal hypo-elastic waves are necessarily either transverse or longitudinal, just as in an isotropic elastic material, and in distinction to an anisotropic elastic material.

When I proposed[1] the theory of hypo-elasticity, I sought [1][2] "a new concept of elastic behavior, mutually exclusive with the theory of finite [elastic] strain except in the linearized case," and I was surprised, although indeed groundlessly, when Noll [2] proved that every isotropic elastic material with invertible stress-strain relation is hypo-elastic. Misled, apparently, by Noll's theorem, Hill and Prager [3] have asserted that every elastic material is hypo-elastic, making hypo-elasticity appear to be a generalization of the classical theory of finite elastic strain. This, certainly, it is not. It embodies a different mathematical concept of the response that in common experience is described as "elastic." Bernstein [4] has set the matter straight by giving a tensorial identity which is a necessary and sufficient condition on the *stress field* in order that solutions of the equations of hypo-elasticity be consistent with some stress-strain relation. As a most illuminating example he has derived the form of the hypo-elastic constitutive equation satisfied by a perfect fluid and has shown that other solutions of that equation are never compatible with the theory of finite elastic strain. His analysis and example make it clear that different assignments of initial stress for the same hypo-elastic constitutive equation lead to solutions representing behavior so diverse that we can scarcely regard them appropriate to any one physical material. Thus hypo-elasticity seems to offer a convenient summary of certain aspects of response common to many materials, rather than a theory of any particular material.

In this note I wish to correct, by clarifying its origin, the misunderstanding growing from the errors of Hill and Prager and then to illustrate further the difference between hypo-elasticity and elasticity by presenting some new results on hypo-elastic waves.

Elasticity is defined by a stress-strain relation of the form

$$T^{km} = \frac{\rho}{\rho_R} x^k{}_{,\alpha} x^m{}_{,\beta} \mathfrak{t}^{\alpha\beta}(\mathbf{C}), \tag{1}$$

where $\mathbf{T}$ is the ordinary Cauchy stress tensor; ρ and ρ_R are the densities in the actual configuration and the reference configuration: $x^k{}_{,\alpha} \equiv \partial x^k / \partial X^\alpha$, $x^k = x^k(\mathbf{X}, t)$ being the deformation; and $\mathfrak{t}$ is a symmetrized function of the components of the symmetric tensor $\mathbf{C}$, where $C_{\alpha\beta} = g_{km} x^k{}_{,\alpha} x^m{}_{,\beta}$. In order that the elastic material be isotropic it is necessary and sufficient[3] that (1) reduce to a relation of the form

$$\mathbf{T} = \mathbf{f}(\mathbf{B}), \tag{2}$$

where $\mathbf{f}$ is an isotropic function and where $\mathbf{B}$ has the components $B^{km} = g^{\alpha\beta} x^k{}_{,\alpha} x^m{}_{,\beta}$. Differentiating (1) materially with respect to time, we easily show that [5]

$$\mathring{T}^{km} + T^{km} D^q_q - T^{kq} \dot{x}^m{}_{,q} - T^{mq} \dot{x}^k{}_{,q} = C^{kmpq} D_{pq}, \tag{3}$$

where $\mathbf{x}$ is the velocity field, $\mathbf{D}$ is the stretching tensor ($D_{km} \equiv x_{(k,m)}$), and

$$C^{kmpq} = 2 \frac{\rho}{\rho_R} x^k{}_{,\alpha} x^m{}_{,\beta} x^p{}_{,\gamma} x^q{}_{,\delta} \frac{\partial \mathfrak{t}^{\alpha\beta}}{\partial C_{\gamma\delta}}. \tag{4}$$

For a given elastic material, the components C^{kmpq} are *given functions of the finite strain and rotation.*

[1] Since wrong statements regarding the origin of hypo-elasticity have been published, its sources will be reviewed here. The basic concept is virtually suggested by Cauchy's theory of initially stressed elastic media (1829), but there is no reference to time rates in his work, which seems to be directed toward infinitesimal static deformation: A.-L. Cauchy, Sur l'équilibre et le mouvement intérieur des corps considérés comme des masses continues, Oeuvres **9**, 243–269. Special theories of hypo-elastic type, employing invariant time rates, were proposed by G. Jaumann in section IX of his Geschlossenes System physikalischer und chemischer Differentialgesetze, Sitzber. Akad. Wiss. Wien (IIa) **120**, 385–530 (1911), and in later papers by him and by E. Lohr. The general theory was mentioned by E. Fromm, eq (53a) of his Stoffgesetze des isotropen Kontinuums, insbesondere bei zähplastischen Verhalten, Ing.-Arch. **4**, 432–466 (1933), as beig appropriate "zur Darstellung gewisser Erscheinungen der Nachwirkung," but he did not develop its properties. A visco-elastic theory including a special case of hypo-elasticity had been proposed earlier by S. Zaremba, Sur une forme perfectionnée de la théorie de la relaxation, Bull. Int. Acad. Sci. Cracovie **1903**, 534–614. However, Zaremba did not mention its relevance for elastic response.

In the course of exposition of some results obtained by Murnaghan under uncertain and dubious assumptions, I proposed a theory only slightly less general than hypo-elasticity in section 56 of the corrections and additions to "The mechanical foundations of elasticity and fluid dynamics," J. Rational Mech. Anal. **2**, 593–616 (1953); at that time I cited the work of Cauchy and Zaremba, and I acknowledged help from Ericksen.

[2] Figures in brackets indicate the literature references at the end of this paper.

[3] Sufficiency is classical; necessity was proved by F. D. Murnaghan, On the energy of deformation of an elastic solid, Proc. Nat. Acad. Sci. USA **14**, 889–891 (1928). Here may lie the source of Prager's error, since he takes (2) as the definition of an elastic material, although in a later passage he obtains for hyperelastic materials a result inconsistent with (2) unless the material is isotropic. The relevant passages occur in sections 1 and 2 of chapter X of Prager's book, cited in footnote 4. It should be noted that Prager's use of the terms "elastic" and "hyperelastic" differs from that of Noll, who introduced them (op. cit.), and of other writers on continuum mechanics. According to Prager's definitions, it can be proved that a hyperelastic material is elastic if and only if it is isotropic. For other writers, hyperelasticity is a special case of elasticity, without exception.

Murnaghan's proof of necessity is unnecessarily elaborate, for the result follows at a glance: since both $\mathbf{T}$ and $\mathbf{B}$ are indifferent, under a rotation $\mathbf{Q}$ they become $\mathbf{QTQ}^T$ and $\mathbf{QBQ}^T$. Hence, if (2) is to hold also in the rotating frame, it is necessary that

$$\mathbf{QTQ}^T = \mathbf{Qf(B)Q}^T = \mathbf{f(QBQ}^T).$$

In other words, $\mathbf{f}$ *must* be an isotropic function.

Hypo-elasticity is defined by an equation of the form [4]

$$\dot{T}^{km} - T^{kq}W^{m}{}_{q} - T^{mq}W^{k}{}_{q} = \mathsf{H}^{kmpq}D_{pq}, \tag{5}$$

where $\mathbf{W}$ is the spin ($W_{km} \equiv \dot{x}_{[k,\,m]}$) and where the components H^{kmpq} are *given functions of the stress*. Moreover, in order that (5) satisfy the principle of material indifference [6], it is necessary and sufficient that $\mathsf{H}^{kmpq}D_{pq}$ be an *isotropic* function of $\mathbf{T}$ and $\mathbf{D}$; that is, if we write $\mathsf{H}(\mathbf{T})[\mathbf{D}]$ for the tensor whose components are $\mathsf{H}^{kmpq}D_{pq}$, then

$$\mathsf{H}(\mathbf{QTQ}^{T})[\mathbf{QDQ}^{T}] = \mathbf{Q}\mathsf{H}(\mathbf{T})[\mathbf{D}]\mathbf{Q}^{T}, \tag{6}$$

identically in the symmetric tensors $\mathbf{T}$ and $\mathbf{D}$ and the orthogonal tensor $\mathbf{Q}$.

Now both (5) and (3) may be written in the form

$$\dot{T}^{km} = \mathsf{D}^{kmpq}\dot{x}_{p,\,q}, \tag{7}$$

but the functional dependence of D is different in the two cases. *In hypo-elasticity, D is a given function of stress; in elasticity, D depends, in general, also on the strain and rotation from a reference configuration.* Elasticity will be included in hypo-elasticity, then, *only* when the dependence of D on strain and rotation turns out to be merely apparent.[5] For an isotropic elastic material, by substituting (2) into (4) it is easy to show that

$$\mathsf{C}^{kmpq} = T^{km}g^{pq} - T^{kq}g^{mp} - T^{mq}g^{kp} + 2\,\frac{\partial T^{km}}{\partial B^{pq}}\,B^{qr}; \tag{8}$$

hence C is a function of $\mathbf{B}$ and $\mathbf{T}$ only. Therefore, if (2) is invertible, C is equal to a function of $\mathbf{T}$ only, and so is D. Thus Noll's theorem follows. However, the common domain is not exhausted by Noll's theorem, since an easy calculation shows that elastic fluids with invertible pressure functions $p(\rho)$ are always hypo-elastic, although for them the stress-strain relations (2) are never invertible.

However, *no material that has a natural state and is anisotropically elastic in small deformation from it is hypo-elastic.* As remarked to me by Noll, this fact is most easily seen by putting $\mathbf{T} = 0$ in (5). From (6) it follows that $\mathsf{H}(\mathbf{0})[\mathbf{D}]$ is an isotropic linear function of $\mathbf{D}$. Therefore, *every* hypo-elastic material responds to infinitesimal strain from an unstressed state like an *isotropic* elastic material.

The domain common to elasticity and hypo-elasticity has thus been nearly delimited. For materials having a natural state, it is no larger than that of isotropic elasticity, at least for small strains; it is no smaller than that of isotropic elasticity with invertible stress-relation; elastic fluids (which of course have neither a natural state nor an invertible stress-strain relation) are included if their pressure functions are invertible.

Bernstein [7] has shown that acceleration waves in a hypo-elastic material, provided the stress be assumed continuous across them, must satisfy a propagation condition of exactly the same form as the elastic one:

$$\mathbf{Q}(\mathbf{n})\mathbf{a} = \rho U^{2}\mathbf{a}; \tag{9}$$

where $\mathbf{a}$ is the amplitude and U is the local speed of propagation, and where

$$Q^{km}(\mathbf{n}) = \mathsf{D}^{kpmq}n_{p}n_{q}. \tag{10}$$

Let us examine the nature of the acoustical tensor $\mathbf{Q}(\mathbf{n})$. Since $\mathsf{H}(\mathbf{T})[\mathbf{D}]$ is an isotropic function of $\mathbf{T}$ and $\mathbf{D}$ as well as a linear function of $\mathbf{D}$, the fourth-order tensor H has a representation of the form [8]

$$\begin{aligned}
\mathsf{H}^{kmpq} = {} & g^{km}(\square_{1}g^{pq} + \square_{2}T^{pq} + \square_{3}T^{pr}T^{q}_{r}) \\
& + T^{km}(\square_{4}g^{pq} + \square_{5}T^{pq} + \square_{6}T^{pr}T^{q}_{r}) \\
& + T^{k}{}_{s}T^{m}_{r}(\square_{7}g^{pq} + \square_{8}T^{pq} + \square_{9}T^{pr}T^{q}_{r}) \\
& + \square_{10}g^{k(p}g^{q)m} \\
& + \frac{1}{2}\,\square_{11}(g^{kp}T^{mq} + g^{mp}T^{kq} + g^{kq}T^{mp} + g^{mq}T^{kp}) \\
& + \frac{1}{2}\,\square_{12}(g^{kp}T^{mr}T^{q}_{r} + g^{mp}T^{kr}T^{q}_{r} + g^{kq}T^{mr}T^{p}_{r} + g^{mq}T^{kr}T^{p}_{r}).
\end{aligned} \tag{11}$$

The coefficients $\square_{\Gamma}$, scalar functions of $\mathbf{T}$, are not uniquely defined, since at most six symmetric second-order tensors can be linearly independent: however, it is not generally justifiable to set any particular $\square_{\Gamma}$ equal to zero. Since, by (5),

$$\mathsf{D}^{kmpq} = \frac{1}{2}\,(T^{kq}g^{pm} - T^{km}g^{pq} + T^{pq}g^{km} - T^{pm}g^{kq}) + \mathsf{H}^{kpmq}, \tag{12}$$

(10) is equivalent to

$$\begin{aligned}
Q^{km}(\mathbf{n}) = \frac{1}{2}\,(g^{km}T^{pq}n_{p}n_{q} - T^{km} + T^{kq}n_{q}n^{m} \\
- T^{mq}n_{q}n^{k}) + \mathsf{H}^{kpmq}n_{p}n_{q}.
\end{aligned} \tag{13}$$

Substitution of (11) into (13) yields an explicit if complicated formula for the acoustical tensor for a general direction of propagation, $\mathbf{n}$, in a general hypo-elastic material.

[4] We use the flux introduced by Zaremba in 1903, op. cit. footnote 2, and explained by Noll, section 7 of op. cit. footnote 3 and T. Y. Thomas, Kinematically preferred co-ordinate systems, Proc. Nat. Acad. Sci. USA 41, 762–770 (1955). A different motivation for use of this flux was given by G. Jaumann, Elektromagnetische Vorgänge in bewegten Medien, Sitzber. Adad. Wiss. Wien (IIa) 15, 337–390 (1906); see sec. IV. As in my own first work, we could equally well use Cauchy's flux, which stands on the left-hand side of (3). In some recent literature these two fluxes have come to be named, for no good reason, after Jaumann and me, respectively. Any properly formulated rate theory is invariant under change of time flux. It is difficult to discern the objective of the recent literature on the merits and demerits of various time fluxes.

[5] Here may be the point where the argument of Hill, cited in reference 3, went astray. Hill employed convected coordinates, which confuse the constant metric of *space* with the changing metric it induces in the *body* in the course of deformation. The role of the rotation becomes obscure when time rates are calculated in such a system. Apparently Hill concluded that if the relation $\mathbf{t} = \mathbf{t}(\mathbf{C})$ is invertible, the coefficient-tensor D in (7) becomes a function of $\mathbf{T}$ alone. From the explicit formula (8), valid in *any* coordinate system, we see that this is not so: Even if $\mathbf{C}$ is determined uniquely by $\mathbf{t}(\mathbf{C})$, the elasticity C still depends on both strain and rotation, for anisotropic materials. It is of course impossible that the strain and rotation can determine a unique stress $\mathbf{T}$ except when $\mathbf{T}$ is in fact independent of the rotation, for a rotation-dependent stress must certainly vanish for *all* rotations if it vanishes for one particular one. More generally and precisely, the principle of material indifferences forces the stress to depend on the rotation in the specific, explicit way implied by (1).

Confusion has been caused by my failure, in my original paper (cited in [1]), to see that (6) must be imposed. However, even with my first and too broad definition, the theorem claimed by Hill and Prager could not be saved, since it reduces to an assertion about the functional *dependence* D in (7) for *elasticity*, irrespective of the invariance properties of H in hypo-elasticity.

In the theory of finite strain of isotropic elastic materials, I have defined *principal waves* as those propagating in the direction of a principal axis of stress [9], and this definition may be adopted also in hypo-elasticity. A theorem I proved for isotropic elastic materials may now be extended to all hypo-elastic materials: *Every principal wave in a hypo-elastic material is either longitudinal or transverse; indeed, the acoustic axes for principal waves are the principal axes of stress.* To prove this, let $\mathbf{n}_1$ be a unit vector parallel to the axis corresponding to the principal stress t_1, so that $T^k_m n^m_1 = t_1 n^k_1$. Then from (13) and (11) we see that

$$\begin{aligned} Q^{km}(\mathbf{n}_1) = \Big[&\square_1 + \frac{1}{2}\square_{10} + (\square_2 + \square_4 + \square_{11})t_1 \\ &+ (\square_3 + \square_5 + \square_7 + \square_{12})t_1^2 \\ &+ (\square_6 + \square_8)t_1^3 + \square_9 t_1^4 \Big] n^k_1 n^m_1 \\ &+ \frac{1}{2}[\square_{10} + (\square_{11} + 1)t_1 + \square_{12}t_1^2]g^{km} \\ &+ \frac{1}{2}(\square_{11} - 1)T^{km} + \frac{1}{2}\square_{12}T^{kr}T^m_r. \end{aligned} \tag{14}$$

The principal axes of this symmetric tensor are the principal axes of stress. The squared speeds of propagation of the corresponding waves may be written down by inspection:

$$\begin{aligned} \rho U^2_{11} &= Q^{km}(\mathbf{n}_1)n^1_k n^1_m, \\ &= \square_1 + \square_{10} + (\square_2 + \square_4 + 2\square_{11})t_1 \\ &\quad + (\square_3 + \square_5 + \square_7 + 2\square_{12})t_1^2 \\ &\quad + (\square_6 + \square_8)t_1^3 + \square_9 t_1^4, \\ \rho U^2_{12} &= Q^{km}(\mathbf{n}_1)n^2_k n^2_m, \\ &= \frac{1}{2}(t_1 - t_2) + \frac{1}{2}\square_{10} + \frac{1}{2}\square_{11}(t_1 + t_2) + \frac{1}{2}\square_{12}(t_1^2 + t_2^2). \end{aligned} \tag{15}$$

These results may be compared with the formulae for the squared speeds of principal waves in isotropic elastic materials [13]. Beyond the identity of the acoustic axes in the two theories, the similarity does not continue. In order to be possible in an isotropic elastic material, the nine squared speeds of propagation, U^2_{ab}, regarded as functions of the principal stretches, must satisfy numerous conditions of compatibility; if they do satisfy them, they determine the form of the stress-strain relation uniquely. The only conditions of compatibility to be satisfied by the hypo-elastic speeds (15), regarded as functions of the principal stresses, are three identities of the form

$$\rho(U^2_{12} - U^2_{21}) = t_1 - t_2, \ldots, \tag{16}$$

in addition, of course, to identities expressing invariance under renumbering of axes. The reciprocal theorem (16), stating that the difference of the speeds of transverse waves when amplitude and wave normal are permuted is twice the corresponding principal shear stress divided by the density, is the only property of elastic principal-wave speeds that carries over, in general, to hypo-elastic materials. One cannot even prove that longitudinal waves travel faster than transverse waves in a state of hydrostatic stress. In the theory of elasticity, knowledge of $\mathbf{Q}(\mathbf{n}_a)$ for the three principal directions $\mathbf{n}_a$ determines $\mathbf{Q}(\mathbf{n})$ for all $\mathbf{n}$ and hence determines also the form of the stress-strain relations. In hypo-elasticity, there is no indication that the three tensors $\mathbf{Q}(\mathbf{n}_a)$ determine the infinitely many tensors $\mathbf{Q}(\mathbf{n})$. Certainly, by (15), knowledge of the nine quantities U^2_{ab} as functions of t_1, t_2, t_3 does not determine the twelve coefficients $\square_\Gamma$ as functions of these same variables, but the question is obscured by the indeterminacy of the $\square_\Gamma$, already remarked.

These results illuminate the nature of hypo-elasticity. It is totally inappropriate for materials regarded physically as "anisotropic." It gives to the physical notion of "elastic isotropy" a mathematical form that is less restrictive than the one embodied in the classical theory of finitely elastic, isotropic materials. In heuristic terms, hypo-elasticity retains the directional aspects of elastic isotropy while relaxing the relations among magnitudes carried along by the notion of isotropy in the theory of finite strain. It is possible that hypo-elasticity may be appropriate to physical materials that show no sign of preferred states, preferred directions, or fading memory.

References

[1] C. Truesdell, Hypo-elasticity, J. Rational Mech. Anal. **4**, 83–133, 1019–1020 (1955), see sec. 3.

[2] W. Noll, On the continuity of the solid and fluid states, J. Rational Mech. Anal. **4**, 3–81 (1955), see sec. 156.

[3] R. Hill, Some basic principles in the mechanics of solids without a natural time, J. Mech. Phys. Solids **7**, 209–225 (1959), writes on p. 220, "Consequently, an elastic solid is hypo-elastic in Truesdell's sense . . . This is a simple proof of a theorem of Noll (1955), stated only for isotropic materials." W. Prager in sec. 1 of ch. X of his Introduction to mechanics of continua, Ginn & Co., 1961 (also Einführung in die Kontinuumsmechanik, Birkhäuser, 1961), writes, "hypo-elastic behavior . . . corresponds to a minimal requirement that a material must fulfill, if it is in any sense to qualify as elastic." The probable sources of these errors are mentioned in footnotes 3 and 5.

[4] B. Bernstein, Hypo-elasticity and elasticity, Arch. Rational Mech. Anal. **6**, 89–104 (1960).

[5] E.g., sec. 55 of my Mechanical Foundations of Elasticity and Fluid Dynamics, J. Rational Mech. Anal. **1**, 125–300 (1952).

[6] E.g., sec. 293 of C. Truesdell and R. Toupin's Classical Field Theories, Flügge's Encyclopedia of Physics **3**$_1$, 226–793 (1960).

[7] B. Bernstein, Conditions for second-order waves in hypo-elasticity, Trans. Soc. Rheol. **6**, 263–273 (1962). Similar results, under assumptions partly more general and partly more special, were obtained by Hill, Acceleration waves in solids, J. Mech. Phys. Solids **10**, 1–16 (1962).

[8] This representation follows by specializing more general results derived by R. S. Rivlin and J. L. Ericksen in sections 40 and 33 of Stress-deformation relations for isotropic materials, J. Rational Mech. Anal. **4**, 323–425 (1955).

[9] C. Truesdell, General and exact theory of waves in finite elastic strain, Arch. Rational Mech. Anal. **8**, 263–296 (1961), see sec. 10.

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Error Bounds in the Pointwise Approximation of Solutions of Elastic Plate Problems

J. H. Bramble* and L. E. Payne*

(May 27, 1963)

Methods are presented for obtaining upper and lower pointwise bounds for the deflection (and its derivatives) in certain physically interesting elastic plate problems.

1. Introduction

In a recent paper [1][1] the authors presented a new method for obtaining pointwise bounds in the first boundary value problem for elastic plates. The method is based on a mean value inequality and an a priori inequality for biharmonic functions. In this paper we use the same techniques to obtain pointwise bounds in terms of known data for the deflection and its derivatives in the second boundary value problem for an elastic plate, i.e., the displacement and bending moment prescribed on the boundary of the plate. In addition we derive pointwise bounds for the solution (and its derivatives) of the third boundary value problem, i.e., the bending moment and reaction prescribed on the plate boundary. In each of these problems bounds at an interior point for the deflection and any derivative up to and including the third are obtained in a straightforward manner. The inequalities obtained, bound an arbitrary (sufficiently smooth) function at a point in a region in terms of the data of the given problem. Thus if one applies the inequality to the difference between a desired solution (with known data) and an arbitrary trial function, it yields a bound for the error in the approximation of the unknown solution by the known trial function. The trial functions need only possess piecewise continuous fourth derivatives and are not required to satisfy any differential equation or boundary conditions. If the applied loading is sufficiently smooth in a neighborhood of the point at which bounds are sought, it is possible to obtain in addition to the above mentioned results, bounds for higher derivatives of the deflection in either of the two problems considered here. We note the somewhat unexpected fact that it is considerably easier to obtain bounds in the third boundary value problem than in either of the first two.

In a subsequent paper the authors will present methods for obtaining pointwise bounds in mixed boundary value problems for elastic plates.

2. Notation and Definitions

Throughout this paper we assume the plate to occupy a two dimensional bounded region R with boundary C. If R is not polygonal it is necessary (for our method) in the second boundary value problem to assume that C have bounded curvature. We make use of the summation convention throughout this paper and employ a comma to denote differentiation, e.g., $W_{,i}=\partial W/\partial x^i$. The symbol σ is used to denote Poisson's ratio and D to signify the plate rigidity. The symbol Δ denotes the Laplace operator and Δ^2 the biharmonic operator. Additional notation will be defined as need arises in the text.

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[1] Figures in brackets indicate the literature references at the end of this paper.

3. Second Boundary Value Problem for Elastic Plates

The boundary value problem considered in this section may be stated as:

$$\Delta^2 u = f \tag{3.1}$$

in R and

$$u = g, M(u) = h \tag{3.2}$$

on C. The quantities f, g, h are prescribed data and $M(u)$ is the bending moment; i.e.,

$$M(u) = -D\left[\Delta u - (1-\sigma)\left(\frac{\partial^2 u}{\partial s^2} + \frac{1}{\rho}\frac{\partial u}{\partial n}\right)\right]. \tag{3.3}$$

In (3.3) $\frac{\partial^2 u}{\partial s^2}$ denotes the second tangential derivative of u, $\partial u/\partial n$ the normal derivative and ρ is the radius of curvature on C.

It is possible to obtain bounds for the strain energy, but since this quantity is not usually of physical interest we do not present a bound for it here. We indicate now a method for obtaining pointwise bounds for u and its derivatives. The result may be stated as follows:

THEOREM 1: *For any biharmonic function* V *with piecewise continuous fourth derivatives in a domain* R *whose boundary* C *has bounded curvature, the following a priori bound holds for any* p*th order derivative of* V *at a point* O *in* R:

$$|V^{(p)}(O)|^2 \leq K_1(p)\oint_C V^2 ds + K_2(p)\oint_C \left(\frac{\partial V}{\partial s}\right)^2 ds + K_3(p)\oint_C \left(\frac{\partial^2 V}{\partial s^2}\right)^2 ds + K_4(p)\oint_C [M(V)]^2 ds, \tag{3.4}$$

where the constants $K_1(p), \ldots, K_4(p)$ are explicitly determined.

Before proving this theorem we indicate how such an a priori inequality leads to pointwise bounds for u and its derivatives in R. If we denote by φ any particular solution of (3.1) and set $V = u - \varphi - \overline{B}$, where $\overline{B}$ is biharmonic, then by choosing the data of $\overline{B}$ to approximate that of $u - \varphi$ (the usual Rayleigh-Ritz technique [2]) we obtain pointwise bounds for u and its derivatives.

In proving theorem I we make use of mean value inequalities derived in [1], i.e., if S_r denotes the interior of a sphere contained in R and of radius r about O, then

$$|V(O)|^2 \leq \frac{4}{\pi r^2}\int_{S_r} V^2 dxdy \leq \frac{4}{\pi r^2}\int_R V^2 dxdy \tag{3.5}$$

$$|V^{(1)}(O)|^2 \leq \frac{36}{\pi r^4}\int_{S_r} V^2 dxdy \leq \frac{36}{\pi r^4}\int_R V^2 dxdy \tag{3.6}$$

with similar expressions for the higher derivatives (see [1]). Thus in order to establish (3.4) it is sufficient to obtain a bound $\int_R V^2 dxdy$ in terms of the quantities on the right of (3.4).

To this end we decompose V as

$$V = B + H \tag{3.7}$$

where

$$\Delta H = O \tag{3.8}$$

in R and

$$H = V \tag{3.9}$$

on C. Then

$$\Delta^2 B = O \tag{3.10}$$

in R and

$$B = O$$
$$M(B) = M(V) - M(H) \tag{3.11}$$

[2] Cf., Handbuch der Physik, I, Mathematische Methoden, Springer Berlin, p. 278 (1956).

on C. Clearly

$$\int_R V^2 dxdy \leq 2\left\{\int_R B^2 dxdy + \int_R H^2 dxdy\right\}. \tag{3.12}$$

We first derive a bound for $\int_R B^2 dxdy$. To do this we introduce an auxiliary function W satisfying

$$\Delta^2 W = B \tag{3.13}$$

in R and

$$W = O, M(W) = O \tag{3.14}$$

on C. Then using the divergence theorem we obtain

$$\int_R B^2 dxdy = \int_R B\Delta^2 W dxdy = -\frac{1}{D}\oint_C M(B)\frac{\partial W}{\partial n} ds. \tag{3.15}$$

Thus by Schwarz's inequality

$$\int_R B^2 dxdy \leq \left[\frac{1}{D^2}\oint_C [M(B)]^2 t^{-1} ds \oint_C t\left(\frac{\partial W}{\partial n}\right)^2 ds\right]^{\frac{1}{2}}, \tag{3.16}$$

where t is any positive function on C.

In order to bound the last integral in (3.16) we introduce a vector field with components f^i such that $f^i n_i > O$ on C. Then setting

$$t = f^i n_i \tag{3.17}$$

we have

$$2\int_R f^i W_{,i}\Delta W dxdy = \oint_C t\left(\frac{\partial W}{\partial n}\right)^2 ds + \int_R (f^i_{,k}\delta^l_i - 2f^l_{,i}) W_{,i} W_{,l} dxdy. \tag{3.18}$$

This is one form of an identity due to Rellich (cf. [5]).

If the derivatives of f^i are bounded in R, we obtain by applying the arithmetic-geometric mean inequality to (3.18)

$$\oint_C t\left(\frac{\partial W}{\partial n}\right)^2 ds \leq \left(c + \frac{b}{\alpha}\right)\int_R W_{,i} W_{,i} dxdy + \alpha\int_R (\Delta W)^2 dxdy. \tag{3.19}$$

Here c is just some bound for the largest eigenvalue of the matrix $2f^l_{,i} - f^k_{,k}\delta^l_i$, i.e.,

$$\int_R (2f^l_{,i} - f^k_{,k}\delta^l_i) W_{,i} W_{,l} dxdy \leq c\int_R W_{,i} W_{,i} dxdy, \tag{3.20}$$

and

$$b = \max (f^i f^i). \tag{3.21}$$

In (3.19), α is any positive constant. For a star shaped boundary we may choose $f^i = x^i$ in which case we may take $c = 0$. Now, since W vanishes on C, it is clear that

$$\int_R W^2 dxdy \leq \frac{1}{\lambda_1}\int_R W_{,i} W_{,i} dxdy \leq \frac{1}{\lambda_1^2}\int_R (\Delta W)^2 dxdy \tag{3.22}$$

where λ_1 is the first eigenvalue of the fixed membrane problem for R.[3]

The strain energy $E(W, W)$ is given by

$$E(W, W) = \frac{D}{2}\left\{\int_R [\sigma(\Delta W)^2 dxdy + (1-\sigma) W_{,ij} W_{,ij}] dxdy\right\}. \tag{3.23}$$

[3] For a discussion of the fixed membrane problem see for instance Courant, R., and Hilbert, D., Methods of Mathematical Physics I, Interscience (1953).

It follows easily, since $W_{,ij}W_{,ij} \geq \frac{1}{2}(\Delta W)^2$, that

$$\int_R (\Delta W)^2 dxdy \leq \frac{2}{1+\sigma}\left[\frac{2}{D}E(W, W)\right], \tag{3.24}$$

and, since W and $M(W)$ vanish on C, $\frac{2}{D}E(W, W) = \int_R W\Delta^2 W dxdy$. Hence

$$\int_R (\Delta W)^2 dxdy \leq \frac{2}{(1+\sigma)}\int_R W\Delta^2 W dxdy. \tag{3.25}$$

From (3.22)

$$\int_R W^2 dxdy \leq \frac{1}{\lambda_1^2}\int_R (\Delta W)^2 dxdy. \tag{3.26}$$

We thus obtain, using Schwarz's inequality on (3.25) and inserting (3.26),

$$\int_R (\Delta W)^2 dxdy \leq \left[\frac{2}{(1+\sigma)\lambda_1}\right]^2 \int_R (\Delta^2 W)^2 dxdy. \tag{3.27}$$

Combining (3.19), (3.22) and (3.26) we now have (choosing $\alpha = [b/\lambda_1]^{1/2}$),

$$\oint_C t\left(\frac{\partial W}{\partial n}\right)^2 ds \leq \frac{4}{(1+\sigma)^2\lambda_1^3}(c + 2[b\lambda_1]^{1/2})\int_R (\Delta^2 W)^2 dxdy. \tag{3.28}$$

In (3.28) we may use any lower bound for λ_1, e.g., that obtained from the isoperimetric inequality of Faber [3] and Krahn [4]

$$\lambda_1 \leq \pi j_0^2/A. \tag{3.29}$$

Here j_0 is the first zero of the Bessel function J_0 and A denotes the area of R. We now set

$$A_1 = \frac{4}{(1+\sigma)^2\bar{\lambda}_1^3}(c + 2[b\bar{\lambda}_1]^{1/2}) \tag{3.30}$$

where $\bar{\lambda}_1$ is any lower bound for λ_1. Then from (3.16) we obtain (recalling (3.13))

$$\int_R B^2 dxdy \leq \frac{A_1}{D^2}\oint_C [M(B)]^2 ds \tag{3.31}$$

From (3.3), (3.9), and (3.11) we have

$$M(B) = M(V) - D(1-\sigma)\left[\frac{\partial^2 V}{\partial s^2} + \frac{1}{\rho}\frac{\partial H}{\partial n}\right].$$

Thus

$$\int_R B^2 dxdy \leq \frac{2A_1}{D^2}\left\{\oint_C \left[M(V) - D(1-\sigma)\frac{\partial^2 V}{\partial s^2}\right]^2 ds + D^2(1-\sigma)^2\oint_C \frac{1}{\rho^2}\left(\frac{\partial H}{\partial n}\right)^2 ds\right\}. \tag{3.32}$$

We assume that the radius of curvature is bounded away from zero on C, i.e.,

$$\rho^2 \geq \beta > 0 \tag{3.33}$$

on C.

In [2, 5] the inequality

$$t_m\oint_C \left(\frac{\partial H}{\partial n}\right)^2 ds \leq \oint_C t^{-1}\left[2f^i\frac{\partial H}{\partial s}\frac{\partial x^i}{\partial s} - cH\right]^2 ds + 2\oint_C t\left(\frac{\partial H}{\partial s}\right)^2 ds$$
$$\leq \oint_C t^{-1}\left[2f^i\frac{\partial V}{\partial s}\frac{\partial x^i}{\partial s} - cV\right]^2 ds + 2\oint_C t\left(\frac{\partial V}{\partial s}\right)^2 ds \tag{3.34}$$

was derived. Here t_m denotes the minimum value of t on C, and c is given by (3.20). Inserting

(3.34) into (3.32) we then obtain (using 3.33) the inequality

$$\int_R B^2 dxdy \leq 2A_1\left\{D^{-2}\oint_C\left[M(V)-D(1-\sigma)\frac{\partial^2 V}{\partial s^2}\right]^2 ds + \frac{(1-\sigma)^2}{\beta t_m}\left[\oint_C t^{-1}\left[2f^i\frac{\partial x^i}{\partial s}\frac{\partial V}{\partial s}-cV\right]^2 ds+2\oint_C t\left(\frac{\partial V}{\partial s}\right)^2 ds\right]\right\}. \tag{3.35}$$

Again, for a star-shaped boundary, we may take $f^i=x^i$ ($x^1=x, x^2=y$) and $c=0$, since $x^i n_i>0$ is just the condition for star-shapedness.

It remains to determine a bound for $\int_R H^2 dxdy$ in (3.12). Such a bound, however, was given in [1]; i.e.,

$$\int_R H^2 dxdy \leq k_2\oint_C H^2 ds = k_2\oint_C V^2 ds \tag{3.36}$$

where k_2 is given by

$$k_2=\frac{1}{\lambda_1}\left[4\lambda_1(f^i f^i)_M+c^2+2c\ \inf\left\{|f^i_{,i}|_M 2\lambda_1[f^i f^i]_M^{1/2}\right\}\right]^{1/2}. \tag{3.37}$$

The symbol $(\ \ )_M$ denotes the maximum value of the quantity in parentheses.
Combining (3.35) and (3.36) and inserting into (3.12) we obtain the inequality

$$\int_R V^2 dxdy \leq 2A_1\left\{2D^{-2}\oint_C\left[M(V)-D(1-\sigma)\frac{\partial^2 V}{\partial s^2}\right]^2 ds+\frac{2(1-\sigma)^2}{\beta t_m}\left[\oint_C t^{-1}\left[f^i\frac{\partial x^i}{\partial s}\frac{\partial V}{\partial s}-CV\right]^2 ds + \oint_C t\left(\frac{\partial V}{\partial s}\right)^2 ds\right]+\frac{k_2}{A_1}\oint_C V^2 ds\right\}. \tag{3.38}$$

The insertion of (3.38) into (3.5), (3.6) or the similar expressions for higher derivatives yields an expression which is not precisely in the form (3.4). However, (3.4) can be easily obtained if the arithmetic-geometric mean inequality is used in (3.38). This completes the proof of theorem I.

In some cases it may be difficult to compute a particular solution φ and its derivatives in R and on C. In that case we may approximate u by any sufficiently smooth function ψ and decompose as follows:

$$u-\psi=V+W \tag{3.39}$$

where

$$\Delta^2 V=O \tag{3.40}$$

in R and

$$V=u-\psi, M(V)=M(u-\psi) \tag{3.41}$$

on C, while

$$\Delta^2 W=\Delta^2(u-\psi) \tag{3.42}$$

in R and

$$W=O, M(W)=O \tag{3.43}$$

on C.

We have already established a bound (3.4) for $V^{(p)}(O)$. We seek now a bound for $W^{(p)}(O)$. Clearly,

$$W(P)=\int_R G(P,P')\Delta^2(u-\psi)dxdy \tag{3.44}$$

where $G(P, P')$ is the Green's function in the simply supported plate problem, i.e., $G(P, P')$ denotes the deflection of a simply supported plate at a point P' in R due to a unit point load at P.[4] We now approximate $W(P)$ by

$$\overline{W}(P)=\int_R[\Gamma(P,P')+\overline{B}]\Delta^2(u-\psi)dxdy, \tag{3.45}$$

[4] For a discussion of G cf. S. Timoshenko, Theory of plates and shells, pp. 156–167 (McGraw-Hill, 1940).

where

$$\Gamma(P,P')=\frac{1}{8\pi}r_{PP'}^2\log\frac{1}{r_{PP'}}, \tag{3.46}$$

the singular part of $G(P,P')$, and $\overline{B}$ denotes any biharmonic function (possibly zero). We approximate a pth derivative, $W^{(p)}(P)$, by $\overline{W}^p(P)$, where

$$\overline{W}^p(P)=\left[\int_R\Gamma(P,P')\Delta^2(u-\psi)dxdy\right]^{(p)}+\int_R\overline{B}^p\Delta^2(u-\psi)dxdy. \tag{3.47}$$

Here $[\quad]^{(p)}$ denotes the pth derivative with respect to the variables of the point P, and $\overline{B}^p$ is an arbitrary biharmonic function, which may be taken identically zero. Clearly then at an interior point P

$$W^{(p)}(P)-\overline{W}^p(P)=\int_R[G(P,P')-\Gamma(P,P')]^{(p)}\Delta^2(u-\psi)dxdy-\int_R\overline{B}^p\Delta^2(u-\psi)dxdy. \tag{3.48}$$

The order of differentiation and integration is interchangeable in the first integral since $G(P,P')-\Gamma(P,P')$ is biharmonic in R. For simplicity we adopt the convention that for a function of the two points P and P', e.g., $\Gamma(P,P')$, the symbol $\Gamma^{(p)}(P,P')$ is to denote differentiation with respect to the variables of the first argument. Thus $\Gamma^{(p)}(P,P')$ denotes differentiation with respect to the variables of the point P while $\Gamma^{(p)}(P',P)$ denotes differentiation with respect to the variables of the point P'.

In view of (3.39) we obtain

$$|u^{(p)}(O)-\psi^{(p)}(O)-\overline{W}^p(O)|\leq|V^{(p)}(O)|+|W^{(p)}(O)-\overline{W}^p(O)|. \tag{3.49}$$

The term $\overline{W}^p(O)$ involves only known quantities and hence may be computed directly. If $p\leq 3$ the differentiation indicated in the first term on the right of (3.47) may be taken under the integral sign and placed on $\Gamma(P,P')$. The resulting integral exists provided $\Delta^2(u-\psi)=O(r_{PP'}^{\epsilon-1})$ as $P\to P'$ for some positive ϵ. If we wish to compute $\overline{W}^p(O)$ for $p\geq 3$ we decompose $\overline{W}^p(P)$ into

$$\overline{W}^p(P)=\overline{W}_1^p(P)+\overline{W}_2^p(P) \tag{3.50}$$

where

$$\overline{W}_1^p(P)=\left[\int_R\Gamma(P,P')\Delta^2(u-\psi)dxdy\right]^{(p)} \tag{3.51}$$

and

$$\overline{W}_2^p(P)=\int_R\overline{B}^p\Delta^2(u-\psi)dxdy. \tag{3.52}$$

Since $\overline{B}^p$ is biharmonic the last term presents no difficulty. In order to evaluate $\overline{W}_1^p(P)$ it is necessary to make some assumptions on the smoothness of $\Delta^2(u-\psi)$. We assume for the moment that $\Delta^2(u-\psi)$ has a sufficient number of derivatives in some spherical neighborhood S_δ of radius δ about the origin and that P lies on the interior of S_δ. Then

$$\begin{aligned}\overline{W}_1^1(P)&=\int_R\Gamma^{(1)}(P,P')\Delta^2(u-\psi)dxdy\\&=\int_{R-S_\delta}\Gamma^{(1)}(P,P')\Delta^2(u-\psi)dxdy-\int_{S_\delta}\Gamma^{(1)}(P',P)\Delta^2(u-\psi)dxdy.\end{aligned} \tag{3.53}$$

We now integrate by parts in S_δ and obtain

$$\begin{aligned}\overline{W}_1^1(P)=&\int_{R-S_\delta}\Gamma^{(1)}(P,P')\Delta^2(u-\psi)dxdy\\&-\int_{\Sigma_\delta}n'\Gamma(P,P')\Delta^2(u-\psi)ds+\int_{S_\delta}\Gamma(P,P')[\Delta^2(u-\psi)]^{(1)}dxdy.\end{aligned} \tag{3.54}$$

Here we have used the fact that $\Gamma(P,P')=\Gamma(P',P)$. In (3.54) Σ_δ denotes the boundary of S_δ and n' the component of the unit normal in the direction in which the differentiation is taken. Since P lies interior to S_δ any derivatives of $\Gamma(P,P')$ with respect to the variables of the point P are bounded on Σ_δ and in $R-S_\delta$. The last integral on the right of (3.54) may be differentiated 3 times with respect to the variables of P and the order of integration and differentiation interchanged provided $[\Delta^2(u-\psi)]^{(1)}=O(r_{PP'}^{\epsilon-1})$ for some positive ϵ as $P'\to P$ in S_δ. Letting $P\to O$ we thus obtain the value of $\overline{W}^4(O)$.

To define $\overline{W}^p(O)$ for $p>4$ we differentiate (3.54), make use of the fact that $\Gamma^{(1)}(P,P')=-\Gamma^{(1)}(P',P)$, integrate the resulting final term by parts, and continue in this manner until we obtain an integral over S_δ which has the desired number of derivatives. We see in this way that for $\overline{W}^p(O)$ to remain finite, it is sufficient that $\Delta^2(u-\psi)$ have $p-4$ derivatives in S_δ and that the derivatives of order $p-3$ be $O(R_{PP'}^{\epsilon-1})$ for some $\epsilon>O$ as $P'\to P$ in S_δ.

We need now only a bound for the last expression on the right of (3.49). To this end we use Schwarz's inequality in (3.48) to obtain

$$[W^{(p)}(O)-\overline{W}^p(O)]^2\leq\int_R\{[G(O,P')-\Gamma(O,P')]^{(p)}-\overline{B}^p\}dxdy\int_R[\Delta^2(u-\psi)]^2dxdy. \quad (3.55)$$

Since the differentiations in (3.55) are with respect to the variable of P (before evaluation at O) the quantity $\{G(O,P')-\Gamma(O,P')]^{(p)}-B^p\}$ is just another function V, i.e., $\Delta^2V=O$ in R, with V and $M(V)$ given on C. Thus (3.38) is valid with V replaced by $\{[G(O,P')-\Gamma(O,P')]^{(p)}-\overline{B}^p\}$ throughout. Since $G^{(p)}(O,P')$ and $M(G^{(p)}(O,P'))$ vanish on C only the quantity $[\Gamma^{(p)}(O,P')+\overline{B}^p]$ appears for V on the right. We note that the right hand side of (3.55) may be made small by choosing $\overline{B}^p$ to approximate the data of $-\Gamma^{(p)}(O,P')$ and/or by choosing ψ to approximate the data of u.

Let us denote the right hand side of (3.38) by $I(V)$ and designate the constant in the mean value inequality for $|V^{(p)}(O)|$ as C_p, i.e.,

$$|V^{(p)}(O)|\leq C_p\int_R V^2dxdy. \quad (3.56)$$

Then our bound may be expressed in the simple form

$$|u^{(p)}(O)-\psi^{(p)}(O)-\overline{W}^{(p)}(O)|\leq C_pI(u-\psi)+I(\Gamma^{(p)}(O,P')+\overline{B}^p)\int_R[\Delta^2(u-\psi)]^2dxdy. \quad (3.57)$$

4. Third Boundary Value Problem for Elastic Plates

In this section we determine a priori bounds for a function V, assuming Δ^2V (the loading) is prescribed throughout a simply connected region R and that the boundary values of $\overline{M}(V)$ and $\overline{Q}(V)$ are prescribed on C. Here $\overline{M}(V)$ is equal to $-M(V)/D$ where D is given by (2.1) and $M(V)$ is given by (3.3). Also

$$\overline{Q}(V)=\frac{\partial}{\partial n}(\Delta V)-(1-\sigma)\left[\frac{\partial^3V}{\partial s^2\partial n}-\frac{\partial}{\partial s}\left(\frac{1}{\rho}\frac{\partial V}{\partial s}\right)\right]. \quad (4.1)$$

The quantities $\overline{M}(V)$ and $\overline{Q}(V)$, are proportional to the bending moment and the reaction. We prove in this section the following theorem.

THEOREM II. *Let* V *be any function with piecewise continuous fourth derivatives in* R *which satisfies the conditions*

$$\oint_C Vds=\oint_C\frac{\partial V}{\partial x}ds=\oint_C\frac{\partial V}{\partial y}ds=O.$$

Then at any point O *in* R,

$$|V^{(p)}(O)|^2 \leq k_1 \oint_C [\overline{M}(V)]^2 ds + k_2 \oint_C [\overline{Q}(V)]^2 ds + k_3 \int_R \eta(\Delta^2 V)^2 dxdy \tag{4.2}$$

where the constants k_i *are explicitly determined,* $p \leq 3$, *and* η *is a known nonnegative function in* R.

Here, as before, $V^{(p)}(x, y)$ denotes any pth order derivative of V. As indicated in the previous section once an a priori inequality of the type (4.2) has been derived it may be used to obtain pointwise bounds in the indicated boundary value problem.

Before proceeding further we define quantities p_2 and q_2 in terms of which the constants in (4.2) will be expressed. These quantities are defined as follows:

$$p_2 = \min_{\oint_C \chi ds = 0} \frac{\int_R \chi_{,i}\chi_{,i} dxdy}{\oint_C \chi^2 ds}, \tag{4.3}$$

where the minimum is taken over all piecewise continuously differentiable functions χ which satisfy the condition $\oint_C \chi ds = O$;

$$q_2 = \min_{\oint_C \psi ds = 0} \frac{\oint_C \left(\frac{\partial \psi}{\partial s}\right)^2 ds}{\oint_C \psi^2 ds} \tag{4.4}$$

where the minimum is taken among all functions ψ continuously differentiable on C and satisfying $\oint_C \psi ds = O$. The eigenvalues corresponding to the Rayleigh quotient in (4.3) are commonly referred to as Stekloff eigenvalues. They are the eigenvalues of the following problem

$$\Delta h = O \tag{4.5}$$

in R and

$$\frac{\partial h}{\partial n} - ph = O \tag{4.6}$$

on C. In general, p_2 will not be known explicitly. However, for our purposes any positive lower bound $\overline{p}_2$ for p_2 will suffice. We indicate later how this lower bound is obtained. The eigenvalues corresponding to the Rayleigh quotient of (4.4) are the Wirtinger eigenvalues. They satisfy

$$\frac{\partial^2 u}{\partial s^2} + qu = O \tag{4.7}$$

on C and the proper continuity condition. The eq (4.7) is of course the equation of the vibrating string of length equal to the perimeter L of C. Thus

$$q_2 = \frac{4\pi^2}{L^2}. \tag{4.8}$$

We proceed now to the proof of theorem II. Let us again denote by $\Gamma(P, P')$ the fundamental solution of the biharmonic equation.

$$\Gamma(P, P') = \frac{1}{8\pi} r_{PP'}^2 \log\left(\frac{1}{r_{PP'}}\right). \tag{4.9}$$

(We could add any biharmonic function B to Γ without altering the following results.) In terms of $\Gamma(P,P')$ we have

$$V(P)=\oint_C V\overline{Q}(\Gamma)ds-\oint_C\frac{\partial V}{\partial n}\overline{M}(\Gamma)ds+\oint_C\overline{M}(V)\frac{\partial\Gamma}{\partial n}ds-\oint_C\overline{Q}(V)\Gamma ds+\int_R\Gamma\Delta^2Vdxdy. \tag{4.10}$$

For P on the interior or R we may differentiate (4.10) with respect to the variables of the point P and interchange order of differentiation and integration on the right. Thus we obtain for $p\leqslant 3$,

$$V^{(p)}(O)=\oint_C V\overline{Q}(\Gamma^{(p)})ds-\oint_C\frac{\partial V}{\partial n}\overline{M}(\Gamma^{(p)})ds+\oint_C\overline{M}(V)\frac{\partial}{\partial n}\Gamma^{(p)}ds-\oint_C\overline{Q}(V)\Gamma^{(p)}ds$$
$$+\int_R\Gamma^{(p)}\Delta^2Vdxdy. \tag{4.11}$$

Here again as in the previous section we could add to $\Gamma^{(p)}$ any biharmonic function B^p. The last three terms of (4.11) involve only the known function Γ and the data of V. Hence we define

$$\Phi^{(p)}(O)=V^{(p)}(O)-\oint_C\overline{M}(V)\frac{\partial\Gamma^{(p)}}{\partial n}ds+\oint_C\overline{Q}(V)\Gamma^{(p)}ds-\int_R\Gamma^{(p)}\Delta^2Vdxdy. \tag{4.12}$$

and obtain by Schwarz's inequality

$$|\Phi^{(p)}(O)|^2\leqslant\left[\left(\frac{2\pi}{L}\right)^2\oint_C V^2ds+\oint_C\left(\frac{\partial V}{\partial n}\right)^2ds\right]\left\{\left(\frac{L}{2\pi}\right)^2\oint_C[\overline{Q}(\Gamma^{(p)})]^2ds+\oint_C[\overline{M}(\Gamma^{(p)})]^2ds\right\}. \tag{4.13}$$

Since $\oint_C Vds=O$ it follows from (4.4) and (4.8) that

$$\oint_C V^2ds\leqslant\left(\frac{L}{2\pi}\right)^2\oint_C\left(\frac{\partial V}{\partial s}\right)^2ds. \tag{4.14}$$

Thus

$$|\Phi^{(p)}(O)|^2\leqslant\left[\oint_C\left(\frac{\partial V}{\partial s}\right)^2ds+\oint_C\left(\frac{\partial V}{\partial n}\right)^2ds\right]\left\{\left(\frac{L}{2\pi}\right)^2\oint_C[\overline{Q}(\Gamma^{(p)})]^2ds+\oint_C[\overline{M}(\Gamma^{(p)})]^2ds\right\}. \tag{4.15}$$

Now

$$\oint_C\left(\frac{\partial V}{\partial s}\right)^2ds+\oint_C\left(\frac{\partial V}{\partial n}\right)^2ds=\oint_C\left[\left(\frac{\partial V}{\partial x}\right)^2+\left(\frac{\partial V}{\partial y}\right)^2\right]ds. \tag{4.16}$$

In view of (4.3) and the fact that $\oint_C\left(\frac{\partial V}{\partial x}\right)ds=\oint_C\left(\frac{\partial V}{\partial y}\right)ds=O$ we have

$$\oint_C\left[\left(\frac{\partial V}{\partial x}\right)^2+\left(\frac{\partial V}{\partial y}\right)^2\right]ds\leqslant\frac{1}{p_2}\int_R(V,_{xi}V,_{xi}+V,_{yi}V,_{yi})dxdy=\frac{1}{p_2}\int_R V,_{ij}V,_{ij}dxdy. \tag{4.17}$$

Thus from (4.14), (4.16), (4.17) and the definition of $E(V,V)$ we have for $\sigma>O$

$$\left[\left(\frac{2\pi}{L}\right)^2\oint_C V^2ds+\oint_C\left(\frac{\partial V}{\partial n}\right)^2\right]ds\leqslant D\frac{2}{p_2(1-\sigma)}E(V,V). \tag{4.18}$$

(For $-\frac{1}{2}<\sigma<O$ the factor $(1-\sigma)$ is replaced by $(1+2\sigma)$.) From the divergence theorem we then obtain

$$\left(\frac{2\pi}{L}\right)^2\oint_C V^2ds+\oint_C\left(\frac{\partial V}{\partial n}\right)^2ds\leqslant\frac{2}{Dp_2(1-\sigma)}\left\{\oint_C\frac{\partial V}{\partial n}\overline{M}(V)ds-\oint_C V\overline{Q}(V)ds\right\}. \tag{4.19}$$

An application of Schwarz's inequality yields

$$\left(\frac{2\pi}{L}\right)^2\oint_C V^2ds+\oint_C\left(\frac{\partial V}{\partial n}\right)^2ds\leq\left[Dp_2^2\frac{2}{(1-\sigma)}\right]^2\left\{\oint_C[\overline{M}(V)]^2ds+\left(\frac{L}{2\pi}\right)^2\oint_C[\overline{Q}(V)]^2ds\right\}. \tag{4.20}$$

Insertion of (4.20) back into (4.13) then gives the desired bound

$$|\Phi^{(p)}(O)|^2\leq\left[\frac{2}{Dp_2(1-\sigma)}\right]^2\left\{\oint_C[\overline{M}(V)]^2ds+\left(\frac{L}{2\pi}\right)^2\oint_C[\overline{Q}(V)]^2ds\right\}$$
$$\left\{\oint_C[\overline{M}(\Gamma^{(p)})]^2ds+\left(\frac{L}{2\pi}\right)^2\oint_C[\overline{Q}(\Gamma^{(p)})]^2ds\right\}. \tag{4.21}$$

This is equivalent to the desired inequality (4.1), since instead of (4.12) we could have used Schwarz's inequality on the last three terms, being careful to use

$$\int_R\Gamma^{(p)}\Delta^2V\,dxdy\leq\int_R r_{pp'}[\Gamma^{(p)}]^2dxdy\int_R r_{pp'}^{-1}\Delta^2V\,dxdy\text{ for }p=3.$$

It remains now to derive a bound for p_2 (i.e., find a $\overline{p}_2$). We denote by u_2 the eigenfunction corresponding to p_2, i.e., (see (4.4) and (4.5))

$$\Delta u_2=O \tag{4.22}$$

in R and

$$\frac{\partial u_2}{\partial n}-p_2u_2=O \tag{4.23}$$

on C. From (4.4) and (4.8) we have

$$\oint_C u_2^2ds\leq\left(\frac{L}{2\pi}\right)^2\oint_C\left(\frac{\partial u_2}{\partial s}\right)^2ds. \tag{4.24}$$

We consider again the identity (see [5])

$$\oint_C t\left(\frac{\partial u_2}{\partial s}\right)^2ds=\oint_C t\left(\frac{\partial u_2}{\partial n}\right)^2ds+2\oint_C f^i\frac{\partial x^i}{\partial s}\frac{\partial u_2}{\partial n}\frac{\partial u_2}{\partial s}ds+\int_R[f^k_{\ ,k}\delta_{il}-2f^i_{\ ,l}]u_{2,\,i}u_{2,\,l}dxdy \tag{4.25}$$

which was used in deriving (3.34). Setting

$$\int_R[f^k_{\ ,k}\delta_{il}-2f^i_{\ ,l}]u_{2,\,i}u_{2,\,l}dxdy\leq c\int_R u_{2,\,i}u_{2,\,i}dxdy \tag{4.26}$$

we obtain from (4.25) (by making use of the arithmetic-geometric mean inequality)

$$t_m\oint_C\left(\frac{\partial u_2}{\partial s}\right)^2ds\leq\oint_C t\left(\frac{\partial u_2}{\partial s}\right)^2ds\leq\bar{b}\oint_C\left(\frac{\partial u_2}{\partial n}\right)^2ds+2c\int_R u_{2,\,i}u_{2,\,i}dxdy, \tag{4.27}$$

where

$$\bar{b}=2\left\{\left[2\left(f^i\frac{\partial x^i}{\partial s}\right)^2+t^2\right]/t\right\}_{\max}\leq 2\{2bt_m^{-1}-t_m\} \tag{4.28}$$

with b given by (3.21). Inserting (4.28) into (4.24) and making use of (4.22) and (4.23) we obtain

$$\oint_C u^2ds\leq\left(\frac{L}{2\pi}\right)^2t_m^{-1}\{p_2^2\bar{b}+2cp_2\}\oint_C u^2ds. \tag{4.29}$$

It follows then that

$$\left(p_2+\frac{c}{\bar{b}}\right)^2 \geq 4\pi^2 t_m(\bar{b}L^2)^{-1}+[c/\bar{b}]^2 \tag{4.30}$$

or that

$$p_2 \geq \bar{p}_2 \equiv [4\pi^2 t_m(\bar{b}L^2)^{-1}+(c/\bar{b})^2]^{1/2}-c/\bar{b}. \tag{4.31}$$

If the boundary C of R is star-shaped with respect to the origin (i.e., $x^i n_i > 0$) then we may choose $f^i = x^i$, in which case $c=0$, $t=x^i n_i$, and $\bar{b}=2\{2r^2{}_M t_m^{-1}-t_m\}$.

Thus far we have indicated how to obtain bounds for the function and its first, second, and third derivatives. In physical problems this includes all derivatives for which we would normally desire bounds. However, if we wish bounds for higher derivatives they may be obtained using the techniques of the previous section. For interior bounds the only term which will cause difficulty for $p > 3$ is the last term on the right of (4.12); but this is precisely the term which was evaluated in the previous section.

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Effect of Error in Measurement of Elastic Constants on the Solutions of Problems in Classical Elasticity

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It is well known that a small error in the measurement of the elastic constants will result, for all physically interesting boundary value problems, in small errors in the computed values of the stresses and displacements. In this paper actual bounds are given for the error in both the first and second boundary value problems. In addition it is shown that as Poisson's ratio tends to ½ the results for compressible theory tend to those for the analogous problems in the classical incompressible theory.

1. Introduction

In the boundary value problems of classical elasticity the elastic constants of the material in question are assumed known. The solution of the problem is then expressed in terms of the "known" elastic constants. However, the values of these elastic constants are actually determined empirically and are therefore not known exactly. Rather the values of these constants may be determined to lie between certain limits.

It appears to be well understood by workers in classical elasticity that a small error in measurement of the elastic constants will not have much effect on the solution to the problem in question, and in problems where the solution can be determined explicitly for arbitrary elastic constants the effect of a small error can be easily determined. However, if the solution cannot be computed but instead pointwise bounds for the solution and its derivatives are sought, one must also take into account the error which might arise due to inaccuracies in the determination of the elastic constants.

In this paper we develop a priori inequalities appropriate to the first and second boundary value problems. These inequalities give upper and lower bounds for the pointwise error in approximation of the exact solution by an arbitrary function. The error in measurement of the elastic constants as well as the error in approximation of the data of the problem is taken into account in the bounds. Thus by choosing trial functions to closely approximate the data, a close approximation can be obtained, with a known bound on the error, provided the error in the constants is also known. The part of the error which results from the fact that the trial function does not satisfy exactly the data of the problem is quadratic. Thus, the Rayleigh-Ritz technique may be employed. This means that we choose from a class of trial functions the best function in the sense that the error term involving the data is minimized (see e.g., [3][1]).

The above mentioned inequalities also are interesting in that they clearly display the fact that the solution to the first (or second) boundary value problem for an incompressible medium (Poisson's ratio, σ, having the value ½) as formulated in the literature, is actually the limit of the solution for values of $\sigma < ½$. We show in fact that at points in a given region the solution for $\sigma < ½$ converges with all its derivatives to the classical solution. To the authors' knowledge this fact has not been pointed out in the literature.

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[1] Figures in brackets indicate the literature references at the end of this paper.

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2. Notation

Let D be a bounded domain with boundary Γ in three dimensions. The displacement vector $u^\alpha(u_1^\alpha, u_2^\alpha, u_3^\alpha)$ corresponding to elastic constants μ_α and σ_α is assumed to satisfy in D the system of equations

$$\Delta u_i^\alpha = p_{,i}^\alpha - \mu_\alpha^{-1} F_i, \qquad i = 1, 2, 3$$

$$u_{j,j}^\alpha = -c_\alpha p^\alpha. \tag{2.1}$$

Here Δ denotes the Laplace operator, the symbol, i, denotes partial differentiation with respect to x_i and the summation convention is employed for Latin but not Greek indices. The constant c_α is given by

$$c_\alpha = 1 - 2\sigma_\alpha, \tag{2.2}$$

and F_i denotes the body force per unit volume. In terms of u_i^α and the elastic constants the stress components τ_{ij}^α are expressed as

$$\tau_{ij}^\alpha = \mu_\alpha [u_{i,j}^\alpha + u_{j,i}^\alpha - (1 - c_\alpha) p^\alpha \delta_{ij}]. \tag{2.3}$$

The physically interesting values of c_α lie in the interval

$$0 \leqslant c_\alpha < 3. \tag{2.4}$$

In this paper we shall, for simplicity, restrict our attention to the first and the second boundary value problems in classical elasticity. In particular, we assume that in D

$$\Delta u_i^\alpha - p_{,i}^\alpha = -\mu_\alpha^{-1} F_i; \qquad \Delta u_i^\beta - p_{,i}^\beta = -\mu_\beta^{-1} F_i$$

$$u_{j,j}^\alpha = -c_\alpha p^\alpha \quad ; \quad u_{j,j}^\beta = -c_\beta p^\beta. \tag{2.5}$$

We wish to compare the solutions u_i^α and u_i^β if one of the following sets of boundary conditions is satisfied:

$$u_i^\alpha = u_i^\beta = f_i \text{ on } \Gamma \tag{2.5A}$$

$$\mu_\alpha^{-1} \tau_{ij}^\alpha n_j = \mu_\beta^{-1} \tau_{ij}^\beta n_j = g_i \text{ on } \Gamma \tag{2.5B}$$

$$\tau_{ij}^\alpha n_j = \tau_{ij}^\beta n_j = h_i \text{ on } \Gamma. \tag{2.5C}$$

From the point of view of actual physical applicability, problem (2.5A) and (2.5C) are the two interesting cases.

In the subsequent sections we shall set

$$w_i = u_i^\alpha - u_i^\beta. \tag{2.6}$$

3. Pointwise Convergence

We demonstrate in this section that if p^β and F_i are square integrable in D and h_i is square integrable on Γ, then at interior points of D, u_i^α and all its derivatives tend to u_i^β and its corresponding derivatives as $c_\alpha \to c_\beta$, and $\mu_\alpha \to \mu_\beta$.

We assume that there exists a constant M_β such that

$$\int_D [p^\beta]^2 dv < M_\beta. \tag{3.1}$$

Let us assume now that we were able to obtain for either of problems (2.5A) to (2.5C), an L_2 bound for w_i in terms of L_2 integrals of the data of u_i^β. Suppose further that the coefficients of the integrals of data tend to zero as $c_\alpha \to c_\beta$ and $\mu_\alpha \to \mu_\beta$. Then if the data of u_i^β are in L_2 it follows that as $\alpha \to \beta$, $w_i \to 0$ in L_2. Pointwise convergence would then follow from the continuity of w_i. The required L_2 bounds for w_i are derived in appendix 1 and appendix 2. Since we are also interested in computing pointwise bounds, explicit constants will be obtained. In problem (2.5A) we have

$$\int_D w_i w_i dv \leq \frac{M_\beta}{2\lambda_1 c_\alpha}(c_\alpha - c_\beta)^2 + \frac{(\mu_\alpha^{-1} - \mu_\beta^{-1})^2}{\lambda_1^2} \int_B F_i F_i dv, \tag{3.2}$$

and in problem (2.5B)

$$\int_D w_i w_i dv \leq \frac{k_4}{2} F(c_\alpha) M_\beta (c_\alpha - c_\beta)^2 + (\mu_\alpha^{-1} - \mu_\beta^{-1})^2 k_4^2 \int_D F_i F_i dv. \tag{3.3}$$

These inequalities follow from (3.1) using (5.23) and (5.30) of appendix 1. For case (2.5C) we have instead from (3.1) and appendix 2,

$$\int_D w_i w_i dv \leq \frac{k_4 (c_\alpha - c_\beta)^2}{(2-3\gamma)[1-(\gamma-1)c_\alpha]} M_\beta + k_4 \left[\frac{k_1}{k_3} \frac{(\mu_\alpha^{-1} - \mu_\beta^{-1})^2}{(2-3\gamma)}\right] \int_\Gamma h_i h_i ds + \frac{8(\mu_\alpha^{-1} - \mu_\beta^{-1})}{(2-3\gamma)^2} k_4^2 \int_D F_i F_i dv, \tag{3.4}$$

where γ satisfies (5.8).

We wish to obtain from these expressions pointwise inequalities in terms of the appropriate sets of data. In order to do this we introduce the function $\Gamma_{ik}^\alpha(P, Q)$, the Kelvin fundamental solution of (2.1); i.e.,

$$\Gamma_{ik}^\alpha(P, Q) = \frac{1}{8\pi(1+c_\alpha)} \left\{ \frac{(1+2c_\alpha)\delta_{ik}}{r_{PQ}} + \frac{[x_i(P) - x_i(Q)][x_k(P) - x_k(Q)]}{r_{PQ}^3} \right\} \tag{3.5}$$

where r_{PQ} is the distance from the point P to the point Q. Then a particular solution $\bar{u}_i^\alpha$ of (2.1) is given by

$$\bar{u}_i^\alpha(P) = \mu_\alpha^{-1} \int_D \Gamma_{ik}^\alpha(P, Q) F_k dv_Q. \tag{3.6}$$

Let u_i^α now be defined as

$$\hat{u}_i^\alpha = u_i^\alpha - \bar{u}_i^\alpha. \tag{3.7}$$

Clearly then $\hat{u}_i^\alpha$ satisfies the system

$$\Delta \hat{u}_i^\alpha + \frac{1}{c_\alpha} \hat{u}_{j,ji}^\alpha = 0 \text{ in } D. \tag{3.8}$$

We now define

$$w_i = \bar{w}_i + \hat{w}_i \tag{3.9}$$

where

$$\overline{w}_i = \overline{u}_i^{\alpha} - \overline{u}_i^{\beta} = \int_D [\mu_{\alpha}^{-1}\Gamma_{ik}^{\alpha}(P,Q) - \mu_{\beta}^{-1}\Gamma_{ik}^{\beta}(P,Q)]F_k dv \tag{3.10}$$

and

$$\hat{w}_i = \hat{u}_i^{\alpha} - \hat{u}_i^{\beta}. \tag{3.11}$$

Since $\hat{u}_i^{\alpha}$ and $\hat{u}_i^{\beta}$ are biharmonic functions, it follows that $\hat{w}_i$ is a biharmonic function. We may, therefore, make use of some mean value inequalities for biharmonic functions (see, e.g., Bramble and Payne [2]). That is, if R denotes the radius of the largest sphere contained in D with center at the point at which bounds are desired (we take this point to be the origin), then

$$\hat{w}_i(0)\hat{w}_i(0) \leq \frac{75}{16\pi R^2}\int_D \hat{w}_i \hat{w}_i dv. \tag{3.12}$$

Similarly, as is shown in [3], it is possible to obtain

$$D^{\nu}(\hat{w}_i)(0)D^{\nu}(\hat{w}_i)(0) \leq C^{\nu}\int_D \hat{w}_i \hat{w}_i dv. \tag{3.13}$$

where D^{ν} is any νth order partial derivative and C^{ν} is a constant depending on R and ν.

It is clear at this point that if in any of problems (2.5A to 2.5C) the body force F_i is identically zero, then $\overline{w}_i \equiv 0$ and $\hat{w}_i \equiv w_i$. By using the appropriate bound for $\int_D w_i w_i dv$ (given by (3.2), (3.3), or (3.4) we would obtain the desired pointwise bounds for $u_i^{\alpha} - u_i^{\beta}$. In fact, in any compact subdomain of D, we could obtain a uniform bound of the following type for any νth order partial derivative, $D^{\nu}(u_i^{\alpha} - u_i^{\beta})$, of $u_i^{\alpha} - u_i^{\beta}$ (assuming that in (3.5), $\int_{\Gamma} h_i h_i ds$ is bounded):

$$D^{\nu}(u_i^{\alpha} - u_i^{\beta})D^{\nu}(u_i^{\alpha} - u_i^{\beta})\,|_{(0,0,0)} \leq k_{\nu}[c_{\alpha} - c_{\beta}]^2 + K_{\nu}[\mu_{\alpha}^{-1} - \mu_{\beta}^{-1}]^2. \tag{3.14}$$

In either of problems (2.5A) or (2.5B), the constant K_{ν} would in fact be zero. We turn now to the case $F_i \not\equiv 0$.

Clearly

$$w_i(0)w_i(0) \leq 2[\overline{w}_i(0)\overline{w}_i(0) + \hat{w}_i(0)\hat{w}_i(0)]. \tag{3.15}$$

But from (3.12)

$$\hat{w}_i(0)\hat{w}_i(0) \leq \frac{75}{8\pi R^3}\left[\int_D w_i w_i dv + \int_D \overline{w}_i \overline{w}_i dv\right]. \tag{3.16}$$

Combining (3.15) and (3.16) we obtain

$$w_i(0)w_i(0) \leq \frac{75}{4\pi R^3}\int_D w_i w_i dv + 2\ \overline{w}_i(0)\overline{w}_i(0) + \frac{75}{4\pi R^3}\int_D \overline{w}_i \overline{w}_i dv. \tag{3.17}$$

We seek now an appropriate bound for the last two terms of (3.17). An application of Schwarz's inequality to (3.10) gives

$$\overline{w}_i \overline{w}_i \leq \int_D [\mu_{\alpha}^{-1}\Gamma_{ik}^{\alpha} - \mu_{\beta}^{-1}\Gamma_{ik}^{\beta}][\mu_{\alpha}^{-1}\Gamma_{ik}^{\alpha} - \mu_{\beta}^{-1}\Gamma_{ik}^{\beta}]dv \int_D F_k F_k dv. \tag{3.18}$$

We now insert (3.5) into the first term on the right and make use of the arithmetic-geometric mean inequality to obtain

$$\int_D [\mu_\alpha^{-1}\Gamma_{ik}^\alpha - \mu_\beta^{-1}\Gamma_{ik}^\beta][\mu_\alpha^{-1}\Gamma_{ik}^\alpha - \mu^{-1}\Gamma_{ik}^\beta]dv \leq \frac{G(P)}{32\pi^2}\left\{2\,[\mu_\alpha^{-1}-\mu_\beta^{-1}]^2 + \left[\frac{(1+2c_\alpha)}{\mu_\alpha(1+c_\alpha)} - \frac{(1+2c_\beta)}{\mu_\beta(1+c_\beta)}\right]^2\right\} \tag{3.19}$$

where $G(P)=\int_D r_{PQ}^{-2}dv_Q$.

Combining (3.18) and (3.19) and integrating we have

$$\int_D \overline{w}_i\overline{w}_i dv \leq \frac{H}{32\pi^2}\left\{2\,[\mu_\alpha^{-1}-\mu_\beta^{-1}]^2 + \left[\frac{(1+2c_\alpha)}{\mu_\alpha(1+c_\alpha)} - \frac{(1+2c_\beta)}{\mu_\beta(1+c_\beta)}\right]^2\right\}\int_D F_k F_k dv, \tag{3.20}$$

where $H=\int_D\int_D r_{PQ}^{-2}dv_Q dv_P$.

It follows now from (3.18), (3.19), and (3.20) that

$$2\overline{w}_i(O)\overline{w}_i(O) + \frac{75}{4\pi R^3}\int_D \overline{w}_i\,\overline{w}_i dv \leq \frac{1}{16\pi^2}\left\{G(O) + \frac{75}{8\pi R^3}H\right\}$$
$$\cdot\left\{2[\mu_\alpha^{-1}-\mu_\beta^{-1}]^2 + \left[\frac{(1+2c_\alpha)}{\mu_\alpha(1+c_\alpha)} - \frac{(1+2c_\beta)}{\mu_\beta(1+c_\beta)}\right]^2\right\}\int_D F_k F_k dv. \tag{3.21}$$

The following bounds on $G(0)$ and H are easy to obtain:

$$G(0) \leq (4\pi d) \tag{3.22}$$

and

$$H \leq V(4\pi d) \tag{3.23}$$

where V is the volume of D and d is the diameter. Thus combining (3.17), and (3.21), (3.22), (3.23), and (3.26) we obtain

$$w_i(0)w_i(0) \leq \frac{75}{4\pi R^3}\int_D w_i w_i dv + \left[\frac{d}{4\pi}\left\{1+\frac{75}{8\pi R^3}V\right\}\right]\cdot$$
$$\cdot\left\{2(\mu_\alpha^{-1}-\mu_\beta^{-1}]^2 + \left[\frac{(1+2c_\alpha)}{\mu_\alpha(1+c_\alpha)} - \frac{(1+2c_\beta)}{\mu_\alpha(1+c_\alpha)}\right]^2\right\}\int_D F_k F_k dv. \tag{3.24}$$

If F_k is square integrable then the insertion of (3.2), (3.3), or (3.5) into (3.24) gives in any compact subdomain D_1 of D a uniform bound of the following type.

$$[u_i^\alpha(P)-u_i^\beta(P)][u_i^\alpha(P)-u_i^\beta(P)] \leq B_1[\mu_\alpha^{-1}-\mu_\beta^{-1}]^2 + B_2[c_\alpha - c_\beta]^2 \tag{3.25}$$

where B_1 and B_2 are independent of the point P in D_1 and can be chosen independent of the elastic constants. This follows because of the fact that since $c_\alpha,\ c_\beta \geq 0$,

$$\left[\frac{(1+2c_\alpha)}{\mu_\alpha(1+c_\alpha)} - \frac{(1+2c_\beta)}{\mu_\beta(1+c_\beta)}\right]^2 = \left\{(\mu_\alpha^{-1}-\mu_\beta^{-1})\,\frac{1+2c_\alpha}{1+c_\alpha} - \mu_\beta^{-1}\,\frac{(c_\alpha-c_\beta)}{(1+c_\alpha)(1+c_\beta)}\right\}^2$$
$$\leq 2\,\{4(\mu_\alpha^{-1}-\mu_\beta^{-1})^2 + \mu_\beta^{-1}(c_\alpha-c_\beta)^2\}. \tag{3.26}$$

The inequality (3.25) clearly implies the uniform convergence of u_i^α to u_i^β in any compact subdomain of D as $\mu_\alpha \to \mu_\beta$ and $c_\alpha \to c_\beta$.

It is clear that if F_k has a sufficient number of square integrable derivatives then the convergence of the derivatives of u_i^α may be obtained in a similar way. To see this we note that

$$\frac{\partial \Gamma_{ik}^{\alpha}(P,Q)}{\partial x_P} \equiv -\frac{\partial \Gamma_{ik}^{\alpha}(P,Q)}{\partial x_Q}. \tag{3.27}$$

Thus (3.6) may be differentiated with respect to the variables of the point P, the order of differentiation and integration being interchanged on the right. Then (3.27) is used followed by an integration by parts which takes the differentiation off of Γ_{ik}^{α}. The bounds for $|u_{i,\rho}^{\alpha}(P)|$ are then obtained in a manner similar to that used in obtaining (3.25). To obtain the desired expressions for higher derivatives, we successively differentiate Γ_{ik}^{α} and use (3.27) to throw the differentiation off of Γ_{ik}^{α} in the volume integral on the right. Then if a sufficient number of derivatives of F_k are square integrable over D and over Γ we obtain a uniform bound similar to (3.25) for any desired derivative of u_i^α. This shows, then, that u_i^α and its derivatives converge to u_i^β and its derivatives.

We note, in particular, that if $c_\beta = 0$ the u_i^β problem is the classical boundary value problem for an incompressible medium. It follows then that as $c_\alpha \to 0$ and $\mu_\alpha \to \mu_\beta$ the solution to the problem corresponding to α converges to the solution in the incompressible case. This fact, to the knowledge of the authors, has never been pointed out in the literature.

Another interesting fact is that F_k need only have the desired number of square integrable derivatives in a neighborhood of the point at which convergence is sought. The procedures used in this section could be applied to any neighborhood Ω of the origin. Then since

$$\int_\Omega w_i w_i dv \leq \int_D w_i w_i dv$$

it would follow that if F_k were square integrable over D and enough derivatives of F_k were square integrable over Ω and its boundary, uniform convergence in Ω of the derivative would follow.

4. Effect of Error in Measurement of Elastic Constants on Pointwise Bounds

We consider now problem (2.1) with either the displacement components or the surface tractions prescribed on Γ. Since c_α and μ_α must be determined empirically we shall assume that some error was made in making the measurements and that the values c_β and μ_β were actually obtained. If φ_i is a sufficiently smooth vector function chosen to approximate the solution u_i^α we may write

$$|u_i^\alpha - \varphi_i| \leq |u_i^\alpha - u_i^\beta| + |u_i^\beta - \varphi_i|. \tag{4.1}$$

We wish to obtain upper and lower bounds for $u_i^\alpha(0)$. If we can make each term on the right of (4.1) less than some prescribed ϵ we would then have

$$-2\epsilon < u_i^\alpha - \varphi_i < 2\epsilon \tag{4.2}$$

and φ_i would be a good pointwise approximation to u_i^α.

The bound for the first term on the right of (4.1) is obtained for the first boundary value problem in elasticity from (3.24) and (5.22) of appendix 1.

$$(u_i^\alpha(0) - u_i^\beta(0))(u_i^\alpha(0) - u_i^\beta(0)) \leq \frac{75}{4\pi R^3}\left\{A_{\alpha\beta}\int_D F_i F_i dv + \frac{3(c_\alpha - c_\beta)^2 \mu_\beta^{-1} E(u^\beta, u^\beta)}{\lambda_1 c_\alpha c_\beta (3 - c_\beta)}\right\}, \tag{4.3}$$

where

$$A_{\alpha\beta} = (1/\lambda_1^2 + 2R^3 d/15 + 5Vd/4\pi)(\mu_\alpha^{-1} - \mu_\beta^{-1})^2 + d\mu_\beta^{-1}(2R^3/75 + V/4\pi)(c_\alpha - c_\beta)^2.$$

A bound for $E(u^\beta, u^\beta)$ is obtained by the methods of Diaz and Greenberg [4], Fichera [6], Synge [10], or Bramble and Payne [3, eq (3.12)].

The second term on the right of (4.1) may be dealt with by the method of Bramble and Payne [3, eq (4.10)].

We look now at the stress components τ_{ij}^α. Using the triangle inequality we obtain

$$|\tau_{ij}^\alpha - \tau_{ij}| \leq |\tau_{ij}^\alpha - \tau_{ij}^\beta| + |\tau_{ij}^\beta - \tau_{ij}|, \tag{4.4}$$

where

$$\tau_{ij} = \mu_\beta \left[\varphi_{i,j} + \varphi_{j,i} + \frac{(1-c_\beta)}{c_\beta} \varphi_{j,j} \delta_{ij} \right] \tag{4.5}$$

is an approximation to τ_{ij}^α. Bounds for the first term on the right of (4.4) may be obtained using (3.13) following again the method outlined at the end of section 3. The second term on the right may be dealt with as described in [3].

For the second boundary value problem in elasticity we again use (4.1) and (4.4), where in this case u_i^α and u_i^β are solutions to (2.5C) and bounds for the second term on the right are given in [1].

5. Appendix 1

In appendix 1 we derive the inequalities (5.22), (5.23), (5.29), and (5.30) which are needed in the text. In order to do this we start with

$$\int_D w_{i,j}(w_{i,j} + w_{j,i}) dv.$$

In particular, we bound this expression in terms of the solution of the problem with elastic constants μ_β, c_β. Consider then the Green's identity

$$\int_D w_{i,j}(w_{i,j} + w_{j,i}) dv = \int_\Gamma w_i (w_{i,j} + w_{j,i}) n_j ds - \int_D w_i (\Delta w_i + w_{j,ji}) ds. \tag{5.1}$$

Making use of (2.5) and integrating by parts we obtain for w_i satisfying (2.6)

$$\int_D w_{i,j}(w_{i,j} + w_{j,i}) dv = -\int_\Gamma w_i \{\mu_\alpha^{-1} \tau_{ij}^\alpha n_j - \mu_\beta^{-1} \tau_{ij}^\beta n_j\} ds + (u_\alpha^{-1} - \mu_\beta^{-1}) \int_D w_i F_i dv$$

$$+ (c_\alpha + c_\beta - 2c_\alpha c_\beta) \int_D p^\alpha p^\beta dv - c_\alpha (1-c_\alpha) \int_D (p^\alpha)^2 dv - c_\beta (1-c_\beta) \int_D (p^\beta)^2 dv. \tag{5.2}$$

In either of problems (2.5A) or (2.5B), the boundary integral on the right of (5.2) will vanish. In fact, this integral would vanish for various mixed boundary conditions which are encountered in physical situations. On the other hand, in problem (2.5C) the boundary term does not vanish. We shall thus consider first problems (2.5A) and (2.5B) and then treat (2.5C) at the end of the section. If $0 < c_\alpha \leq 3/4$ ($\sigma \geq 1/8$) we rewrite (5.2) for (A) and (B) as

$$\int_D w_{i,j}(w_{i,j} + w_{j,i}) dv = -c_\alpha (1-c_\alpha) \int_D \left[p^\alpha - \frac{(c_\alpha + c_\beta - 2c_\alpha c_\beta)}{2c_\alpha (1-c_\alpha)} p^\beta \right]^2 dv$$

$$+ (\mu_\alpha^{-1} - \mu_\alpha^{-1}) \int_D w_i F_i dv + \frac{(c_\alpha - c_\beta)^2}{4c_\alpha (1-c_\alpha)} \int_D (p^\beta)^2 dv. \tag{5.3}$$

The first term on the right is nonpositive. Thus

$$\int_D w_{i,j}(w_{i,j} + w_{j,i}) dv \leq \frac{(c_\alpha - c_\beta)^2}{4c_\alpha (1-c_\alpha)} \int_D (p^\beta)^2 dv + (\mu_\alpha^{-1} - \mu_\beta^{-1}) \int_D w_i F_i dv. \tag{5.4}$$

The strain energy $E(u^\beta, u^\beta)$ corresponding to u^β is given by:

$$2\mu_\beta^{-1}E(u^\beta, u^\beta) = \int_D [u^\beta_{i,j}(u^\beta_{i,j} + u^\beta_{j,i}) + c_\beta(1-c_\beta)(p^\beta)^2]dv. \geq \frac{c_\beta}{3}(3-c_\beta)\int_D (p^\beta)^2 dv. \tag{5.5}$$

The last inequality results from the fact that

$$u^\beta_{i,j}(u^\beta_{i,j} + u^\beta_{j,i}) \leq 2[(u^\beta_{1,1})^2 + (u^\beta_{2,2})^2 + (u^\beta_{3,3})^2] \geq 2/3[u^\beta_{i,i}]^2. \tag{5.6}$$

Hence if $c_\beta \neq 0$ we have from (5.4) and (5.5)

$$\int_D w_{i,j}(w_{i,j} + w_{j,i})dv \geq \frac{3\mu_\beta^{-1}(c_\alpha - c_\beta)^2}{2c_\alpha c_\beta(1-c_\alpha)(3-c_\beta)}E(u^\beta, u^\beta) + (\mu_\alpha^{-1} - \mu_\beta^{-1})\int_D w_i F_i dv. \tag{5.7}$$

Bounds for $E(u^\beta, u^\beta)$ in problems (2.5A) and (2.5B) in terms of the data of the problem have been given for instance by Diaz and Greenberg [4], Fichera [6], Prager and Synge [9], Synge [10], and Bramble and Payne [1, 3]. If $F_i \equiv 0$ such bounds would give then for $0 < c_\alpha \leq 3/4$ an upper bound for

$$\int_D w_{i,j}(w_{i,j} + w_{j,i})dv$$

in terms of the data of the u_i^β problem of (2.5A) or (2.5B).

For $c_\alpha > 3/4$ we choose a constant γ in the interval

$$1 - c_\alpha^{-1} < \gamma < 2/3 \tag{5.8}$$

and consider the following identity derived from (5.2) by addition of the appropriate term to each side of the equation

$$\int_D w_{i,j}(w_{i,j} + w_{j,i})dv - \gamma\int_D w^2_{j,j}dv = [c_\alpha + c_\beta + 2(\gamma-1)c_\alpha c_\beta]\int_D p^\alpha p^\beta dv$$

$$- c_\alpha\{1 + (\gamma-1)c_\alpha\}\int_D (p^\alpha)^2 dv - c_\beta\{1 - (\gamma-1)c_\beta\}\int_D (p^\beta)^2 dv + (\mu_\alpha^{-1} - \mu_\beta^{-1})\int_D w_i F_i dv. \tag{5.9}$$

Using (5.6) then we have

$$\left(1 - \frac{3\gamma}{2}\right)\int_D w_{i,j}(w_{i,j} + w_{j,i})dv \leq -c_\alpha\{1 + (\gamma-1)c_\alpha\}\int_D \left\{p^\alpha + \frac{[c_\alpha + c_\beta + 2(\gamma-1)c_\alpha c_\beta]}{2c_\alpha[1 + (\gamma-1)c_\alpha]}p^\beta\right\}^2 dv$$

$$+ \frac{(c_\alpha - c_\beta)^2}{4c_\alpha[1 + (\gamma-1)c_\alpha]}\int_D (p^\beta)^2 dv + (\mu_\alpha^{-1} - \mu_\beta^{-1})\int_D w_i F_i dv. \tag{5.10}$$

Thus we obtain

$$\int_D w_{i,j}(w_{i,j} + w_{j,i})dv \leq \frac{(c_\alpha - c_\beta)^2}{2(2-3\gamma)c_\alpha[1 + (\gamma-1)c_\alpha]}\int_D [p^\beta]^2 dv + \frac{2(\mu_\alpha^{-1} - \mu_\beta^{-1})}{(2-3\gamma)}\int_D w_i F_i dv. \tag{5.11}$$

The choice $\gamma = 2c_\alpha/(3 + 2c_\alpha)$ yields

$$\int_D w_{i,j}(w_{i,j} + w_{j,i})dv \leq \frac{(3 + 2c_\alpha)^2(c_\alpha - c_\beta)^2}{4c_\alpha(3 - c_\alpha)^2}\int_D (p^\beta)^2 dv + (\mu_\alpha^{-1} - \mu_\beta^{-1})\int_D w_i F_i dv. \tag{5.12}$$

Proceeding as before we obtain finally the inequality

$$\int_D w_{i,j}(w_{i,j} + w_{i,i})dv \leq F(c_\alpha)(c_\alpha - c_\beta)^2\int_D (p^\beta)^2 dv + (\mu_\alpha^{-1} - \mu_\beta^{-1})\int_D F_i w_i dv, \tag{5.13}$$

where

$$F(c_\alpha)=\begin{cases}[4c_\alpha(1-c_\alpha)]^{-1}, 0<c_\alpha\leq 3/4\\ [(3+2c_\alpha)^2[4c_\alpha(3-c_\alpha)^2]^{-1}, c_\alpha>3/4.\end{cases} \tag{5.14}$$

It follows then that for $c_\beta\neq 0$

$$\int_D w_{i,j}(w_{i,j}+w_{j,i})dv\leq\frac{6\mu_\beta^{-1}F(c_\alpha)}{c_\beta(3-c_\beta)}(c_\alpha-c_\beta)^2E(u^\beta,u^\beta)+(\mu_\alpha^{-1}-\mu_\beta^{-1})\int_D w_iF_idv. \tag{5.15}$$

In problem (2.5A) we have the relation

$$\begin{aligned}\int_D w_{i,j}w_{i,j}dv&=-\int_D w_i\Delta w_idv=-\int_D w_i(p^\alpha-p^\beta)dv\\ &=-\int_D(c_\alpha p^\alpha-c_\beta p^\beta)(p^\alpha-p^\beta)dv+(\mu_\alpha^{-1}-\mu_\beta^{-1})\int_D F_iw_idv.\end{aligned} \tag{5.16}$$

or

$$\int_D w_{i,j}w_{i,j}dv=-c_\alpha\int_D\left[p^\alpha-\frac{(c_\alpha+c_\beta)}{2c_\beta}p^\beta\right]^2dv+(\mu_\alpha^{-1}-\mu_\beta^{-1})\int_D F_iw_idv+\frac{(c_\alpha-c_\beta)^2}{4c_\alpha}\int_D(p^\beta)^2dv. \tag{5.17}$$

Thus, dropping the first term on the right,

$$\int_D w_{i,j}w_{i,j}dv\leq\frac{(c_\alpha-c_\beta)^2}{4c_\alpha}\int_D(p^\beta)^2dv+(\mu_\alpha^{-1}-\mu_\beta^{-1})\int_D F_iw_idv \tag{5.18}$$

and if $c_\beta\neq 0$

$$\int_D w_{i,j}w_{i,j}dv\leq 3\mu_\beta^{-1}\frac{(c_\alpha-c_\beta)^2}{2c_\alpha c_\beta(3-c_\beta)}E(\mu^\beta,\mu^\beta)+(\mu_\alpha^{-1}-\mu_\beta^{-1})\int_D F_iw_idv. \tag{5.19}$$

We have retained the term $(\mu_\alpha^{-1}-\mu_\beta^{-1})\int_D F_iw_idv$ up to this point since in the important case $F_i\equiv 0$ this term will vanish and the inequality will not be altered. Let us assume now that $F_i\not\equiv 0$ and note that

$$\int_D w_iw_idv\leq\frac{1}{\lambda_1}\int_D w_{i,j}w_{i,j}dv\leq\frac{3(c_\alpha-c_\beta)^2\mu_\beta^{-1}E(u_\beta,u_\beta)}{2\lambda_1c_\alpha c_\beta(3-c_\beta)}+\frac{(\mu_\alpha^{-1}-\mu_\beta^{-1})}{\lambda_1}\int_D F_iw_idv. \tag{5.20}$$

Where λ_1 is the first eigenvalue in the three dimensional fixed membrane problem for D. A lower bound for λ_1 is given by the Faber-Krahn inequality [5, 7], i.e.,

$$\lambda_1\geq\pi^2\left[\frac{4\pi}{3V}\right]^{2/3}, \tag{5.21}$$

where V denotes the volume of D. From the arithmetic-geometric mean inequality applied to the last term of (5.20) it follows that

$$\int_D w_iw_idv\leq\frac{3(c_\alpha-c_\beta)^2\mu_\beta^{-1}}{\lambda_1c_\alpha c_\beta(3-c_\beta)}E(u_\beta,u_\beta)+\frac{(\mu_\alpha^{-1}-\mu_\beta^{-1})^2}{\lambda_1^2}\int_D F_iF_idv. \tag{5.22}$$

Similarly if we insert (5.18) into the first inequality in (5.20) and apply the arithmetic-geometric mean inequality to the last term we obtain

$$\int_D w_iw_idv\leq\frac{(c_\alpha-c_\beta)^2}{2\lambda_1c_\alpha}\int_D(p^\beta)^2dv+\frac{(\mu_\alpha^{-1}-\mu_\beta^{-1})^2}{\lambda_1^2}\int_D F_iF_idv. \tag{5.23}$$

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For problem (3.5B) it was shown in [1, eq (3.14)] that if w_i is normalized so that $\int_\Gamma w_i ds = 0$ then

$$\int_\Gamma w_i w_i ds \leq \frac{k_1}{2k_3} \int_D w_{i,j}(w_{i,j} + w_{j,i}) dv \tag{5.24}$$

where k_1 and k_3 are constants given in [1, eqs (2.9) and (2.19)], which are computable in terms of the geometry of D.

We now make use of the identity

$$5\int_D w_i w_i dv = \int_\Gamma x^k w_i (2n_i w_k + n_k w_i) ds + 2\int_D x^k w_i (w_{i,k} + w_{k,i}) dv + 2\int_D x^k w_k w_{i,i} dv. \tag{5.25}$$

An application of the arithmetic geometric mean inequality and (3.8) then yields (see e.g., [1, eqs (2.26) to (2.28)])

$$\int_D w_i w_i dv \leq \frac{6r_M}{5} \int_\Gamma w_i w_i ds + \frac{28r_M^2}{25} \int_D w_{i,j}(w_{i,j} + w_{j,i}) dv \tag{5.26}$$

where r_M denotes the maximum distance from the origin to Γ. From (5.24) and (5.26) it follows then that

$$\int_D w_i w_i dv \leq k_4 \int_D w_{i,j}(w_{i,j} + w_{j,i}) dv \tag{5.27}$$

where

$$k_4 = \left[\frac{3}{5} r_M k_1 / k_3 + \frac{28}{25} r_M^2 \right]. \tag{5.28}$$

Thus if in (2.5B) $c_\beta \neq 0$,

$$\int_D w_i w_i dv \leq \frac{12 k_4 \mu_\beta^{-1} F(c_\alpha - c_\beta)^2}{c_\beta (3 - c_\beta)} E(u^\beta, u^\beta) + (\mu_\alpha^{-1} - \mu_\beta^{-1})^2 k_4^2 \int_D F_i F_i dv. \tag{5.29}$$

Finally, after inserting (5.13) into (5.27) and using the arithmetic-geometric mean inequality on the last term we have

$$\int_D w_i w_i dv \leq 2k_4 F(c_\alpha)(c_\alpha - c_\beta)^2 \int_D (p^\beta)^2 dv + k_4^2 (\mu_\alpha^{-1} - \mu_\beta^{-1})^2 \int_D F_i F_i dv. \tag{5.30}$$

6. Appendix 2

We derive here inequalities (6.3) and (6.4) which are needed to treat problem (2.5C).

Again we start with $\int_D w_{i,j}(w_{i,j} + w_{j,i}) dv$ and instead of (5.2) we obtain

$$\int_D w_{i,j}(w_{i,j} + w_{j,i}) dv = -\int_\Gamma w_i h_i [\mu_\alpha^{-1} - \mu_\beta^{-1}] ds + (c_\alpha + c_\beta - 2c_\alpha c_\beta) \int_D p^\alpha p^\beta dv$$

$$- c_\alpha (1 - c_\alpha) \int_D (p^\alpha)^2 dv - c_\beta (1 - c_\beta) \int_D (p^\beta)^2 dv + (\mu_\alpha^{-1} - \mu_\beta^{-1}) \int_D w_i F_i dv, \tag{6.1}$$

which follows from a calculation involving the use of the divergence theorem. We thus obtain as before

$$\int_D w_{i,j}(w_{i,j} + w_{j,i}) dv + \frac{2}{(2 - 3\gamma)} [\mu_\alpha^{-1} - \mu_\beta^{-1}] \int_\Gamma w_i h_i ds$$

$$- \frac{2}{(2 - 3\gamma)} (\mu_\alpha^{-1} - \mu_\beta^{-1}) \int_D F_i w_i dv \leq \frac{(c_\alpha - c_\beta)^2}{2(2 - 3\gamma)[1 + (\gamma - 1)c_\alpha]} \int_D (p^\beta)^2 dv \tag{6.2}$$

where γ satisfies (5.8). It follows then from the arithmetic-geometric mean inequality, (5.24), and (5.27) that if w_i is normalized so that $\int_\Gamma w_i ds = 0$ then

$$\left[1-\frac{a}{2}-\frac{b}{2}\right]\int_D w_{i,j}(w_{i,j}+w_{j,i})dv \leq \frac{[\mu_\alpha^{-1}-\mu_\beta^{-1}]^2}{4(2-3\gamma)^2 a}(k_1/k_3)^2\int_\Gamma h_i h_i ds$$

$$+\frac{(c_\alpha-c_\beta)^2\int_D (p^\beta)^2 dv}{2(2-3\gamma)[1+(\gamma-1)c_\alpha]}+\frac{2(\mu_\alpha^{-1}-\mu_\beta^{-1})^2}{(2-3\gamma)^2 b}k^4\int_D F_i F_i dv \qquad (6.3)$$

for any positive numbers a and b. In particular for $a=\frac{1}{2}$, $b=\frac{1}{2}$ (6.2) becomes

$$\int_D w_{i,j}(w_{i,j}+w_{j,i})dv \leq \left[\frac{k_1}{k_3}\frac{(\mu_\alpha^{-1}-\mu_\beta^{-1})}{(2-3\gamma)}\right]^2\int_\Gamma h_i h_i ds$$

$$+\frac{8(\mu_\alpha^{-1}-\mu_\beta^{-1})^2}{(2-3\gamma)^2}k_4\int_D F_i F_i dv+\frac{(c_\alpha-c_\beta)^2}{(2-3\gamma)[1+(\gamma-1)c_\alpha]}\int_D (p^\beta)^2 dv. \qquad (6.4)$$

As before inserting (6.3) into (5.27) we have

$$\int_D w_i w_i dv \leq k_4\left[\frac{k_1}{k_3}\frac{(\mu_\alpha^{-1}-\mu_\beta^{-1})}{(2-3\gamma)}\right]^2\int_\Gamma h_i h_i ds+\frac{8(\mu_\alpha^{-1}-\mu_\beta^{-1})^2}{(2-3\gamma)^2}\int_D F_i F_i dv$$

$$+\frac{k_4(c_\alpha-c_\beta)^2}{(2-3\gamma)[1+(\gamma-1)c_\alpha]}\int_D (p^\beta)^2 dv. \qquad (6.5)$$

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Eigenfunctions of the f^3 Configuration

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Wave functions for three equivalent f electrons which are eigenfunctions of L^2, S^2, J^2, and J_z are given.

As a preliminary step in some recent theoretical work on Nd^{3+} and Er^{3+} a set of three-electron wave functions was obtained which are eigenfunctions of L^2, S^2, J^2, and J_z. These functions are linear combinations of products of one-electron functions which have the usual form

$$f(r)Y_3^m(\theta, \phi)\,\chi_{m_s}.$$

Y_3^m is a normalized spherical harmonic and χ_{m_s} is a spin eigenfunction.

The three-electron eigenfunctions may be useful to other research workers so they are tabulated here. A possible use for them is in calculating matrix elements or transition probabilities.

To understand the notation used for the wave functions consider the following example: the ket

$$|\,\bar{3}\,2-2> \equiv |\,\bar{3}>|\,2>|-2>$$

is to be understood as a product of one-electron functions. For the first electron m_l is 3 and m_s is $-\frac{1}{2}$; for the second electron m_l is 2 and m_s is $+\frac{1}{2}$; for the third electron m_l is -2 and m_s is $+\frac{1}{2}$.

The functions given in the tables can be antisymmetrized in an obvious way, but this antisymmetrization is usually not necessary in calculations.

The phases of the functions are adjusted so that matrix elements of the spin orbit interaction have the signs given by Judd and Loudon.[1]

There are two $^2H_{11/2}$, $^2H_{9/2}$, $^2G_{9/2}$, $^2G_{7/2}$, $^2F_{7/2}$, $^2F_{5/2}$, $^2D_{5/2}$, and $^2D_{3/2}$ terms. These terms are labeled by additional quantum numbers which refer to group theoretical properties of the wave functions.[2]

Half of the 364 wave functions are given here. The remaining functions can be obtained by reversing the signs of m_l and m_s in the kets.

The tabulated functions can also be used for the system of 11 equivalent f electrons.

TABLE 1. $M = 17/2$

	$N^{-1/2}$	$\lvert 3\bar{3}2>$
$^2L_{17/2}$	1	1

TABLE 2. $M = 15/2$

	$N^{-1/2}$	$\lvert 3\bar{3}\bar{2}>$	$\lvert 3\bar{3}1>$	$\lvert 32\bar{2}>$	$\lvert 321>$
$^2L_{17/2}$	17	1	$\sqrt{10}$	$-\sqrt{6}$	0
$^2L_{15/2}$	136	$8\sqrt{2}$	$-\sqrt{5}$	$\sqrt{3}$	0
$^2K_{15/2}$	8	0	$\sqrt{3}$	$\sqrt{5}$	0
$^4I_{15/2}$	1	0	0	0	-1

TABLE 3. $M = 13/2$

	$N^{-1/2}$	$3\bar{3}\bar{1}$	$3\bar{3}0$	$32\bar{1}$	320	$3\bar{2}1$	$\bar{3}2\bar{2}$	$\bar{3}21$
$^2L_{17/2}$	136	$\sqrt{10}$	$\sqrt{30}$	$-\sqrt{15}$	0	$2\sqrt{15}$	$-\sqrt{6}$	$-\sqrt{15}$
$^2L_{15/2}$	136	$5\sqrt{3}$	-2	$\sqrt{2}$	0	$-2\sqrt{2}$	$-3\sqrt{5}$	$\sqrt{2}$
$^2K_{15/2}$	120	$\sqrt{3}$	6	$5\sqrt{2}$	0	$-2\sqrt{2}$	$\sqrt{5}$	$-3\sqrt{2}$
$^4I_{15/2}$	15	0	0	-1	$-2\sqrt{3}$	-1	0	-1
$^2K_{13/2}$	840	$7\sqrt{6}$	$-3\sqrt{2}$	-5	0	2	$7\sqrt{10}$	3
$^4I_{13/2}$	15	0	0	-2	$\sqrt{3}$	-2	0	-2
$^2I_{13/2}$	21	0	3	$-\sqrt{2}$	0	$-\sqrt{2}$	0	$2\sqrt{2}$

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TABLE 4. $M = 11/2$

	N^{-2}	$\lvert\bar{3}\bar{3}\bar{0}\rangle$	$\lvert\bar{3}\bar{3}-\bar{1}\rangle$	$\lvert\bar{3}\bar{2}\bar{0}\rangle$	$\lvert 32-\bar{1}\rangle$	$\lvert\bar{3}\bar{2}\bar{1}\rangle$	$\lvert\bar{3}\bar{2}0\rangle$	$\lvert 31\bar{1}\rangle$	$\lvert 310\rangle$	$\lvert\bar{3}\bar{2}\bar{1}\rangle$	$\lvert\bar{3}\bar{2}0\rangle$	$\lvert\bar{3}\bar{2}\bar{1}\rangle$	$\lvert\bar{2}\bar{2}\bar{1}\rangle$
$^2L_{17/2}$	136	$\sqrt{6}$	$2\sqrt{2}$	-2	0	$\sqrt{3}$	6	$-\sqrt{30}$	0	$-2\sqrt{3}$	-4	$\sqrt{3}$	$3\sqrt{2}$
$^2L_{15/2}$	952	14	$-2\sqrt{3}$	$\sqrt{6}$	0	$7\sqrt{2}$	$-3\sqrt{6}$	$3\sqrt{5}$	0	$-14\sqrt{2}$	$2\sqrt{6}$	$7\sqrt{2}$	$-3\sqrt{3}$
$^2K_{15/2}$	840	6	$6\sqrt{3}$	$5\sqrt{6}$	0	$3\sqrt{2}$	$\sqrt{6}$	$7\sqrt{5}$	0	$2\sqrt{2}$	$-6\sqrt{6}$	$-5\sqrt{2}$	$\sqrt{3}$
$^4I_{15/2}$	105	0	0	$-2\sqrt{3}$	-6	-1	$-2\sqrt{3}$	0	$-\sqrt{30}$	-1	$-2\sqrt{3}$	-1	0
$^2K_{13/2}$	10920	$39\sqrt{2}$	$-6\sqrt{6}$	$-10\sqrt{3}$	0	39	$-2\sqrt{3}$	$-7\sqrt{10}$	0	26	$12\sqrt{3}$	-65	$-\sqrt{6}$
$^4I_{13/2}$	195	0	0	$-3\sqrt{3}$	6	-4	$-3\sqrt{3}$	0	$\sqrt{30}$	-4	$-3\sqrt{3}$	-4	0
$^2I_{13/2}$	273	3	$6\sqrt{3}$	$-2\sqrt{6}$	0	$-2\sqrt{2}$	$\sqrt{6}$	0	0	$\sqrt{2}$	$\sqrt{6}$	$\sqrt{2}$	$-6\sqrt{3}$
$^4I_{11/2}$	3003	0	0	$11\sqrt{2}$	$-3\sqrt{6}$	$-11\sqrt{6}$	$11\sqrt{2}$	0	$-3\sqrt{5}$	$-11\sqrt{6}$	$11\sqrt{2}$	$-11\sqrt{6}$	0
$^2I_{11/2}$	546	$6\sqrt{6}$	$-3\sqrt{2}$	2	0	$-8\sqrt{3}$	-1	0	0	$4\sqrt{3}$	-1	$4\sqrt{3}$	$3\sqrt{2}$
$^2H_{11/2}$ (210x21)	273	0	$\sqrt{5}$	$3\sqrt{10}$	0	0	$-2\sqrt{10}$	$-6\sqrt{3}$	0	0	$-\sqrt{10}$	0	$-2\sqrt{5}$
$^2H_{11/2}$ (210x11)	6	0	$\sqrt{2}$	0	0	0	-1	0	0	0	1	0	$\sqrt{2}$
$^4G_{11/2}$	11	0	0	0	$\sqrt{5}$	0	0	0	$-\sqrt{6}$	0	0	0	0

TABLE 5. $M = 9/2$

	N^{-2}	$\lvert\bar{3}\bar{3}-\bar{1}\rangle$	$\lvert\bar{3}\bar{3}-2\rangle$	$\lvert 32-\bar{1}\rangle$	$\lvert 32-2\rangle$	$\lvert\bar{3}\bar{2}\bar{0}\rangle$	$\lvert\bar{3}\bar{2}-1\rangle$	$\lvert\bar{3}\bar{1}\bar{0}\rangle$	$\lvert 31-1\rangle$	$\lvert\bar{3}\bar{1}0\rangle$	$\lvert\bar{3}\bar{2}\bar{0}\rangle$	$\lvert\bar{3}2-1\rangle$	$\lvert\bar{3}\bar{2}\bar{1}\rangle$	$\lvert\bar{3}\bar{2}\bar{0}\rangle$	$\lvert\bar{3}\bar{1}\bar{1}\rangle$	$\lvert\bar{3}\bar{1}0\rangle$	$\lvert\bar{2}\bar{2}\bar{1}\rangle$	$\lvert\bar{2}\bar{2}0\rangle$	$\lvert 21\bar{1}\rangle$	$\lvert 210\rangle$
$^2L_{17/2}$	476	$2\sqrt{2}$	$\sqrt{5}$	$-\sqrt{3}$	0	4	$4\sqrt{3}$	$-2\sqrt{10}$	0	$3\sqrt{10}$	-6	$-3\sqrt{3}$	0	2	$-\sqrt{30}$	$-\sqrt{10}$	$3\sqrt{2}$	$4\sqrt{6}$	$-3\sqrt{5}$	0
$^2L_{15/2}$	12376	26	$-2\sqrt{10}$	$2\sqrt{6}$	0	$26\sqrt{2}$	$-8\sqrt{6}$	$8\sqrt{5}$	0	$-12\sqrt{5}$	$-39\sqrt{2}$	$6\sqrt{6}$	0	$13\sqrt{2}$	$-13\sqrt{15}$	$4\sqrt{5}$	39	$-16\sqrt{3}$	$6\sqrt{10}$	0
$^2K_{15/2}$	3640	$6\sqrt{3}$	$2\sqrt{30}$	$10\sqrt{2}$	0	$6\sqrt{6}$	$8\sqrt{2}$	$8\sqrt{15}$	0	$-4\sqrt{15}$	$-\sqrt{6}$	$-18\sqrt{2}$	0	$-5\sqrt{6}$	$7\sqrt{5}$	$-4\sqrt{15}$	$\sqrt{3}$	16	$2\sqrt{30}$	0
$^4I_{15/2}$	455	0	0	-6	$-2\sqrt{10}$	$-2\sqrt{3}$	-6	$-\sqrt{30}$	$-4\sqrt{10}$	$-\sqrt{30}$	$-2\sqrt{3}$	-6	-1	$-2\sqrt{3}$	0	$-\sqrt{30}$	0	0	0	$-2\sqrt{5}$
$^2K_{13/2}$	1820	$6\sqrt{6}$	$-\sqrt{15}$	-5	0	$12\sqrt{3}$	-4	$-2\sqrt{30}$	0	$\sqrt{30}$	$-2\sqrt{3}$	9	0	$-10\sqrt{3}$	$7\sqrt{10}$	$\sqrt{30}$	$\sqrt{6}$	$-4\sqrt{2}$	$-\sqrt{15}$	0
$^4I_{13/2}$	390	0	0	$-2\sqrt{3}$	$\sqrt{30}$	-7	$-2\sqrt{3}$	$-\sqrt{10}$	$2\sqrt{30}$	$-\sqrt{10}$	-7	$-2\sqrt{3}$	$-2\sqrt{3}$	-7	0	$-\sqrt{10}$	0	0	0	$\sqrt{15}$
$^2I_{13/2}$	546	6	$3\sqrt{10}$	$-2\sqrt{6}$	0	$-\sqrt{2}$	$4\sqrt{6}$	$-2\sqrt{5}$	0	$\sqrt{5}$	$-\sqrt{2}$	$-2\sqrt{6}$	0	$2\sqrt{2}$	0	$\sqrt{5}$	-6	$-6\sqrt{3}$	$3\sqrt{10}$	0
$^4I_{11/2}$	33033	0	0	$19\sqrt{6}$	$-6\sqrt{15}$	$-44\sqrt{2}$	$19\sqrt{6}$	$19\sqrt{5}$	$-12\sqrt{15}$	$19\sqrt{5}$	$-44\sqrt{2}$	$19\sqrt{6}$	$-33\sqrt{6}$	$-44\sqrt{2}$	0	$19\sqrt{5}$	0	0	0	$-3\sqrt{30}$
$^2I_{11/2}$	6006	$33\sqrt{2}$	$-6\sqrt{5}$	$4\sqrt{3}$	0	-11	$-8\sqrt{3}$	$2\sqrt{10}$	0	$-\sqrt{10}$	-11	$4\sqrt{3}$	0	22	0	$-\sqrt{10}$	$-33\sqrt{2}$	$6\sqrt{6}$	$-6\sqrt{5}$	0
$^2H_{11/2}$(210x21)	3003	$\sqrt{5}$	$5\sqrt{2}$	$6\sqrt{30}$	0	$\sqrt{10}$	$-3\sqrt{30}$	-6	0	16	$2\sqrt{10}$	$-3\sqrt{30}$	0	$-3\sqrt{10}$	$-6\sqrt{3}$	-10	$-2\sqrt{5}$	$-6\sqrt{15}$	$-8\sqrt{2}$	0
$^2H_{11/2}$(210x11)	66	$\sqrt{2}$	$2\sqrt{5}$	0	0	-1	0	0	0	$-\sqrt{10}$	1	0	0	0	0	$\sqrt{10}$	$\sqrt{2}$	0	$-2\sqrt{5}$	0
$^4G_{11/2}$	121	0	0	$\sqrt{5}$	$5\sqrt{2}$	0	$\sqrt{5}$	$-\sqrt{6}$	$-\sqrt{2}$	$-\sqrt{6}$	0	$\sqrt{5}$	0	0	0	$-\sqrt{6}$	0	0	0	-6
$^4I_{9/2}$	9438	0	0	$2\sqrt{15}$	$-\sqrt{6}$	$-11\sqrt{5}$	$2\sqrt{15}$	$5\sqrt{2}$	$-2\sqrt{6}$	$5\sqrt{2}$	$-11\sqrt{5}$	$2\sqrt{15}$	$22\sqrt{15}$	$-11\sqrt{5}$	0	$5\sqrt{2}$	0	0	0	$-\sqrt{3}$
$^2H_{9/2}$(210x21)	15015	$-5\sqrt{10}$	5	$6\sqrt{15}$	0	$-10\sqrt{5}$	$-3\sqrt{15}$	$-3\sqrt{2}$	0	$8\sqrt{2}$	$-20\sqrt{5}$	$-3\sqrt{15}$	0	$30\sqrt{5}$	$30\sqrt{6}$	$-5\sqrt{2}$	$10\sqrt{10}$	$-3\sqrt{30}$	-8	0
$^2H_{9/2}$(210x11)	66	$-2\sqrt{5}$	$\sqrt{2}$	0	0	$\sqrt{10}$	0	0	0	-1	$-\sqrt{10}$	0	0	0	0	1	$-2\sqrt{5}$	0	$-\sqrt{2}$	0
$^4G_{9/2}$	1452	0	0	$-4\sqrt{10}$	15	0	$-4\sqrt{10}$	$8\sqrt{3}$	-3	$8\sqrt{3}$	0	$-4\sqrt{10}$	0	0	0	$8\sqrt{3}$	0	0	0	$-9\sqrt{2}$
$^4G_{9/2}$(210x21)	455	0	$-5\sqrt{3}$	$3\sqrt{5}$	0	0	$\sqrt{5}$	$-3\sqrt{6}$	0	$-2\sqrt{6}$	0	$-4\sqrt{5}$	0	0	0	$5\sqrt{6}$	0	$\sqrt{10}$	$2\sqrt{3}$	0
$^4G_{9/2}$(210x20)	231	0	$-3\sqrt{6}$	$-\sqrt{10}$	0	0	$2\sqrt{10}$	$2\sqrt{3}$	0	$-\sqrt{3}$	0	$-\sqrt{10}$	0	0	0	$-\sqrt{3}$	0	$-3\sqrt{5}$	$-3\sqrt{6}$	0
$^4F_{9/2}$	4	0	0	0	-1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	$-\sqrt{2}$

TABLE 6. $M = 7/2$

	N^{-2}	$\bar{33}-\bar{2}$	$\bar{33}-3$	$32-\bar{2}$	$32-3$	$\bar{32}-\bar{1}$	$\bar{32}-2$	$31-\bar{1}$	$31-2$	$\bar{31}\bar{0}$	$\bar{31}-1$	$30\bar{0}$
$^2L_{17/2}$	6188	5	$\sqrt{6}$	$-\sqrt{6}$	0	$3\sqrt{15}$	$5\sqrt{6}$	$-5\sqrt{6}$	0	$5\sqrt{2}$	$10\sqrt{6}$	$-10\sqrt{6}$
$^2L_{15/2}$	12376	$2\sqrt{30}$	$-\sqrt{5}$	$\sqrt{5}$	0	$18\sqrt{2}$	$-5\sqrt{5}$	$5\sqrt{5}$	0	$4\sqrt{15}$	$-10\sqrt{5}$	$10\sqrt{5}$
$^2K_{13/2}$	10920	$2\sqrt{30}$	$3\sqrt{5}$	$5\sqrt{5}$	0	$18\sqrt{2}$	$7\sqrt{5}$	$17\sqrt{5}$	0	$4\sqrt{15}$	$-2\sqrt{5}$	$18\sqrt{5}$
$^4I_{13/2}$	1365	0	0	$-2\sqrt{10}$	$-\sqrt{15}$	-6	$-2\sqrt{10}$	$-4\sqrt{10}$	-15	$-\sqrt{30}$	$-4\sqrt{10}$	0
$^2K_{13/2}$	60060	$11\sqrt{15}$	$-3\sqrt{10}$	$-5\sqrt{10}$	0	99	$-7\sqrt{10}$	$-17\sqrt{10}$	0	$11\sqrt{30}$	$2\sqrt{10}$	$-18\sqrt{10}$
$^4I_{13/2}$	4290	0	0	$-\sqrt{10}$	$2\sqrt{15}$	-18	$-\sqrt{10}$	$-2\sqrt{10}$	30	$-3\sqrt{30}$	$-2\sqrt{10}$	0
$^2I_{13/2}$	6006	$3\sqrt{30}$	$6\sqrt{5}$	$-4\sqrt{5}$	0	$6\sqrt{2}$	$14\sqrt{5}$	$-8\sqrt{5}$	0	$-\sqrt{15}$	$10\sqrt{5}$	$-6\sqrt{5}$
$^4I_{11/2}$	33033	0	0	$16\sqrt{3}$	$-9\sqrt{2}$	$-5\sqrt{30}$	$16\sqrt{3}$	$32\sqrt{3}$	$-9\sqrt{30}$	-25	$32\sqrt{3}$	0
$^2I_{11/2}$	6006	30	$-3\sqrt{6}$	$2\sqrt{6}$	0	$4\sqrt{15}$	$-7\sqrt{6}$	$4\sqrt{6}$	0	$-5\sqrt{2}$	$-5\sqrt{6}$	$3\sqrt{6}$
$^2H_{11/2}(210\times21$	15015	$5\sqrt{2}$	$5\sqrt{3}$	$30\sqrt{3}$	0	$3\sqrt{30}$	$-10\sqrt{3}$	$24\sqrt{3}$	0	10	$\sqrt{3}$	$-22\sqrt{3}$
$^2H_{11/2}(210\times11)$	66	2	$\sqrt{6}$	0	0	0	$\sqrt{6}$	0	0	$-\sqrt{2}$	$-\sqrt{6}$	$\sqrt{6}$
$^4G_{11/2}$	605	0	0	$5\sqrt{2}$	$5\sqrt{3}$	$\sqrt{5}$	$5\sqrt{2}$	$-\sqrt{2}$	$4\sqrt{5}$	$-\sqrt{6}$	$-\sqrt{2}$	0
$^4I_{9/2}$	9438	0	0	$3\sqrt{6}$	-2	$-6\sqrt{15}$	$3\sqrt{6}$	$6\sqrt{6}$	$-2\sqrt{15}$	$-15\sqrt{2}$	$6\sqrt{6}$	0
$^2H_{9/2}(210\times21)$	45045	$-15\sqrt{3}$	$5\sqrt{2}$	$30\sqrt{2}$	0	$-27\sqrt{5}$	$-10\sqrt{2}$	$24\sqrt{2}$	0	$-15\sqrt{6}$	$\sqrt{2}$	$-22\sqrt{2}$
$^2H_{9/2}(210\times11)$	198	$-3\sqrt{6}$	2	0	0	0	2	0	0	$3\sqrt{3}$	-2	2
$^4G_{9/2}$	13068	0	0	-25	$15\sqrt{6}$	$-8\sqrt{10}$	-25	5	$12\sqrt{10}$	$16\sqrt{3}$	5	0
$^2G_{9/2}(210\times21)$	4095	$-5\sqrt{3}$	$-15\sqrt{2}$	$15\sqrt{2}$	0	$4\sqrt{5}$	$-10\sqrt{2}$	$-3\sqrt{2}$	0	$-5\sqrt{6}$	$-7\sqrt{2}$	$-6\sqrt{2}$
$^2G_{9/2}(210\times20)$	2079	$-3\sqrt{6}$	-18	-10	0	$\sqrt{10}$	2	2	0	$\sqrt{3}$	14	18
$^4F_{9/2}$	36	0	0	-1	$-\sqrt{6}$	0	-1	1	0	0	1	0
$^4G_{7/2}$	41580	0	0	$-35\sqrt{2}$	$15\sqrt{3}$	$28\sqrt{5}$	$-35\sqrt{2}$	$7\sqrt{2}$	$28\sqrt{5}$	$-28\sqrt{6}$	$7\sqrt{2}$	0
$^2G_{7/2}(210\times21)$	16380	$20\sqrt{6}$	-15	15	0	$-16\sqrt{10}$	-10	-3	0	$40\sqrt{3}$	-7	-6
$^2G_{7/2}(210\times20)$	4158	$12\sqrt{6}$	-9	-5	0	$-4\sqrt{10}$	1	1	0	$-4\sqrt{3}$	7	9
$^4F_{7/2}$	36	0	0	$\sqrt{2}$	$-\sqrt{3}$	0	$\sqrt{2}$	$-\sqrt{2}$	0	0	$-\sqrt{2}$	0
$^2F_{7/2}(210\times21)$	132	0	-7	3	0	0	2	-3	0	0	1	2
$^2F_{7/2}(100\times10)$	6	0	1	1	0	0	-1	-1	0	0	1	1
$^4D_{7/2}$	21	0	0	0	$\sqrt{10}$	0	0	0	$-\sqrt{6}$	0	0	0

TABLE 6. M = 7/2 – *Continued.*

	$30-1$	$\bar{3}2-\bar{1}$	$\bar{3}2-2$	$\bar{3}\bar{2}0$	$\bar{3}\bar{2}-1$	$\bar{3}1\bar{0}$	$\bar{3}1-1$	$\bar{3}\bar{1}0$	$2\bar{2}\bar{0}$	$2\bar{2}$-1	$21\bar{0}$	$21-1$	$\bar{2}1\bar{0}$	$\bar{2}1\bar{1}$	$\bar{2}10$
$^2I_{-17/2}$	0	$-4\sqrt{15}$	$-4\sqrt{6}$	0	$\sqrt{15}$	$-15\sqrt{2}$	$-5\sqrt{6}$	$10\sqrt{2}$	$4\sqrt{30}$	$9\sqrt{10}$	$-10\sqrt{3}$	0	$20\sqrt{3}$	-15	$-10\sqrt{3}$
$^2I_{-15/2}$	0	$-24\sqrt{2}$	$4\sqrt{5}$	0	$6\sqrt{2}$	$-12\sqrt{15}$	$5\sqrt{5}$	$8\sqrt{15}$	48	$-15\sqrt{3}$	$5\sqrt{10}$	0	$-10\sqrt{10}$	$-6\sqrt{30}$	$5\sqrt{10}$
$^2K_{13/2}$	0	$-8\sqrt{2}$	$-12\sqrt{5}$	0	$-10\sqrt{2}$	$4\sqrt{15}$	$-15\sqrt{5}$	$-8\sqrt{15}$	16	$21\sqrt{3}$	$9\sqrt{10}$	0	$-2\sqrt{10}$	$2\sqrt{30}$	$-7\sqrt{10}$
$^4I_{13/2}$	$-2\sqrt{30}$	-6	$-2\sqrt{10}$	$-2\sqrt{3}$	-6	$-\sqrt{30}$	$-4\sqrt{10}$	$-\sqrt{30}$	0	0	$-2\sqrt{5}$	$-3\sqrt{15}$	$-2\sqrt{5}$	0	$-2\sqrt{5}$
$^2K_{13/2}$	0	-44	$12\sqrt{10}$	0	-55	$11\sqrt{30}$	$15\sqrt{10}$	$-22\sqrt{30}$	$44\sqrt{2}$	$-21\sqrt{6}$	$-18\sqrt{5}$	0	$4\sqrt{5}$	$11\sqrt{15}$	$14\sqrt{5}$
$^4I_{13/2}$	$4\sqrt{30}$	-18	$-\sqrt{10}$	$-11\sqrt{3}$	-18	$-3\sqrt{30}$	$-2\sqrt{10}$	$-3\sqrt{30}$	0	0	$-\sqrt{5}$	$6\sqrt{15}$	$-\sqrt{5}$	0	$-\sqrt{5}$
$^2I_{13/2}$	0	$-12\sqrt{2}$	$-10\sqrt{5}$	0	$6\sqrt{2}$	$-\sqrt{15}$	$-2\sqrt{5}$	$2\sqrt{15}$	-18	0	$4\sqrt{10}$	0	$-11\sqrt{10}$	$3\sqrt{30}$	$7\sqrt{10}$
$^4I_{11/2}$	-36	$-5\sqrt{30}$	$16\sqrt{3}$	$-33\sqrt{10}$	$-5\sqrt{30}$	-25	$32\sqrt{3}$	-25	0	0	$8\sqrt{6}$	$-27\sqrt{2}$	$8\sqrt{6}$	0	$8\sqrt{6}$
$^2I_{11/2}$	0	$-8\sqrt{15}$	$5\sqrt{6}$	0	$4\sqrt{15}$	$-5\sqrt{2}$	$\sqrt{6}$	$10\sqrt{2}$	$-6\sqrt{30}$	0	$-4\sqrt{3}$	0	$11\sqrt{3}$	30	$-7\sqrt{3}$
$^2H_{11/2}(210\times21)$	0	$3\sqrt{30}$	$-20\sqrt{3}$	0	$-6\sqrt{30}$	-16	$-25\sqrt{3}$	6	$-6\sqrt{15}$	$-18\sqrt{5}$	$-11\sqrt{6}$	0	$\sqrt{6}$	$-8\sqrt{2}$	$10\sqrt{6}$
$^2H_{11/2}(210\times11)$	0	0	$-\sqrt{6}$	0	0	$\sqrt{2}$	$\sqrt{6}$	0	0	0	$-2\sqrt{3}$	0	$\sqrt{3}$	-2	$\sqrt{3}$
$^4G_{11/2}$	$-\sqrt{6}$	$\sqrt{5}$	$5\sqrt{2}$	0	$\sqrt{5}$	$-\sqrt{6}$	$-\sqrt{2}$	$-\sqrt{6}$	0	0	-6	$-7\sqrt{3}$	-6	0	-6
$^4I_{9/2}$	$-4\sqrt{2}$	$-6\sqrt{15}$	$3\sqrt{6}$	$33\sqrt{5}$	$-6\sqrt{15}$	$-15\sqrt{2}$	$6\sqrt{6}$	$-15\sqrt{2}$	0	0	$3\sqrt{3}$	-6	$3\sqrt{3}$	0	$3\sqrt{3}$
$^2H_{9/2}(210\times21)$	0	$-27\sqrt{5}$	$-20\sqrt{2}$	0	$54\sqrt{5}$	$24\sqrt{6}$	$-25\sqrt{2}$	$-9\sqrt{6}$	$27\sqrt{10}$	$-6\sqrt{30}$	-22	0	2	$24\sqrt{3}$	20
$^2H_{9/2}(210\times11)$	0	0	-2	0	0	$-3\sqrt{3}$	2	0	0	0	$-2\sqrt{2}$	0	$\sqrt{2}$	$3\sqrt{6}$	$\sqrt{2}$
$^4G_{9/2}$	$-6\sqrt{3}$	$-8\sqrt{10}$	-25	0	$-8\sqrt{10}$	$16\sqrt{3}$	5	$16\sqrt{3}$	0	0	$15\sqrt{2}$	$-21\sqrt{6}$	$15\sqrt{2}$	0	$15\sqrt{2}$
$^2G_{9/2}(210\times21)$	0	$-\sqrt{5}$	$-5\sqrt{2}$	0	$-3\sqrt{5}$	$2\sqrt{6}$	$10\sqrt{2}$	$3\sqrt{6}$	$\sqrt{10}$	$7\sqrt{30}$	-6	0	-14	$2\sqrt{3}$	20
$^2G_{9/2}(210\times20)$	0	$-2\sqrt{10}$	8	0	$\sqrt{10}$	$\sqrt{3}$	-16	$-2\sqrt{3}$	$-3\sqrt{5}$	0	$-12\sqrt{2}$	0	0	$-3\sqrt{6}$	$12\sqrt{2}$
$^4F_{9/2}$	$2\sqrt{3}$	0	-1	0	0	0	1	0	0	0	$-\sqrt{2}$	$-\sqrt{6}$	$-\sqrt{2}$	0	$-\sqrt{2}$
$^4G_{7/2}$	$-3\sqrt{6}$	$28\sqrt{5}$	$-35\sqrt{2}$	0	$28\sqrt{5}$	$-28\sqrt{6}$	$7\sqrt{2}$	$-28\sqrt{6}$	0	0	42	$-21\sqrt{3}$	42	0	42
$^2G_{7/2}(210\times21)$	0	$4\sqrt{10}$	-5	0	$12\sqrt{10}$	$-16\sqrt{3}$	10	$-24\sqrt{3}$	$-8\sqrt{5}$	$7\sqrt{15}$	$-3\sqrt{2}$	0	$-7\sqrt{2}$	$-8\sqrt{6}$	$10\sqrt{2}$
$^2G_{7/2}(210\times20)$	0	$8\sqrt{10}$	4	0	$-4\sqrt{10}$	$-4\sqrt{3}$	-8	$8\sqrt{3}$	$12\sqrt{5}$	0	$-6\sqrt{2}$	0	0	$12\sqrt{6}$	$6\sqrt{2}$
$^4F_{7/2}$	$\sqrt{6}$	0	$\sqrt{2}$	0	0	0	$-\sqrt{2}$	0	0	0	2	$-\sqrt{3}$	2	0	2
$^2F_{7/2}(210\times21)$	0	0	-5	0	0	0	2	0	0	$-\sqrt{15}$	$\sqrt{2}$	0	$\sqrt{2}$	0	$-2\sqrt{2}$
$^2F_{7/2}(100\times10)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$^4D_{7/2}$	$\sqrt{5}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0

TABLE 7. $M = 5/2$

	N^2	$3\bar{3}-\bar{3}$	$32-\bar{3}$	$3\bar{2}-\bar{2}$	$3\bar{2}-3$	$31-\bar{2}$	$31-3$	$3\bar{1}-\bar{1}$	$3\bar{1}-2$	$30-\bar{1}$	$30-2$	$\bar{3}0-1$	$\bar{3}2-\bar{2}$	$\bar{3}2-3$	$\bar{3}\bar{2}-\bar{1}$	$\bar{3}\bar{2}-2$
$^2L_{17/2}$	12376	$\sqrt{6}$	-1	$4\sqrt{6}$	6	$-2\sqrt{15}$	0	$5\sqrt{6}$	$5\sqrt{15}$	$-15\sqrt{2}$	0	$20\sqrt{2}$	$-5\sqrt{6}$	-5	0	$\sqrt{6}$
$^2L_{15/2}$	136136	11	$\sqrt{6}$	44	$-6\sqrt{6}$	$6\sqrt{10}$	0	55	$-15\sqrt{10}$	$30\sqrt{3}$	0	$-40\sqrt{3}$	-55	$5\sqrt{6}$	0	11
$^2K_{15/2}$	24024	$3\sqrt{5}$	$\sqrt{30}$	$12\sqrt{5}$	$2\sqrt{30}$	$22\sqrt{2}$	0	$15\sqrt{5}$	$5\sqrt{2}$	$14\sqrt{15}$	0	$-8\sqrt{15}$	$-7\sqrt{5}$	$-3\sqrt{30}$	0	$-5\sqrt{5}$
$^4I_{15/2}$	3003	0	$-\sqrt{15}$	$-2\sqrt{10}$	$-\sqrt{15}$	-15	$-4\sqrt{6}$	$-4\sqrt{10}$	-15	$-2\sqrt{30}$	$-10\sqrt{3}$	$-2\sqrt{30}$	$-2\sqrt{10}$	$-\sqrt{15}$	-6	$-2\sqrt{10}$
$^2K_{13/2}$	24024	$3\sqrt{10}$	$-\sqrt{15}$	$12\sqrt{10}$	$-2\sqrt{15}$	-22	0	$15\sqrt{10}$	-5	$-7\sqrt{30}$	0	$4\sqrt{30}$	$-7\sqrt{10}$	$3\sqrt{15}$	0	$-5\sqrt{10}$
$^4I_{13/2}$	429	0	0	$-\sqrt{10}$	0	0	$2\sqrt{6}$	$-2\sqrt{10}$	0	0	$5\sqrt{3}$	0	$-\sqrt{10}$	0	-6	$-\sqrt{10}$
$^2I_{13/2}$	3003	6	$-\sqrt{6}$	10	$5\sqrt{6}$	$-3\sqrt{10}$	0	2	$6\sqrt{10}$	$-7\sqrt{3}$	0	$8\sqrt{3}$	-14	$-4\sqrt{6}$	0	4
$^4I_{11/2}$	99099	0	$13\sqrt{2}$	$-6\sqrt{3}$	$13\sqrt{2}$	$13\sqrt{30}$	$-24\sqrt{5}$	$-12\sqrt{3}$	$13\sqrt{30}$	52	$-30\sqrt{10}$	52	$-6\sqrt{3}$	$13\sqrt{2}$	$-27\sqrt{30}$	$-6\sqrt{3}$
$^2I_{11/2}$	18018	$9\sqrt{6}$	4	$15\sqrt{6}$	-20	$4\sqrt{15}$	0	$3\sqrt{6}$	$-8\sqrt{15}$	$14\sqrt{2}$	0	$-16\sqrt{2}$	$-21\sqrt{6}$	16	0	$6\sqrt{6}$
$^2H_{11/2}(210x21)$	45045	$5\sqrt{3}$	$30\sqrt{2}$	$20\sqrt{3}$	$-5\sqrt{2}$	$18\sqrt{30}$	0	$25\sqrt{3}$	$-3\sqrt{30}$	4	0	46	$10\sqrt{3}$	$-25\sqrt{2}$	0	$-30\sqrt{3}$
$^2H_{11/2}(210x11)$	198	$\sqrt{6}$	0	$\sqrt{6}$	4	0	0	$-\sqrt{6}$	0	$2\sqrt{2}$	0	$-4\sqrt{2}$	$-\sqrt{6}$	-4	0	0
$^4G_{11/2}$	1815	0	$5\sqrt{3}$	$5\sqrt{2}$	$5\sqrt{3}$	$4\sqrt{5}$	$3\sqrt{30}$	$-\sqrt{2}$	$4\sqrt{5}$	$-\sqrt{6}$	$2\sqrt{15}$	$-\sqrt{6}$	$5\sqrt{2}$	$5\sqrt{3}$	$\sqrt{5}$	$5\sqrt{2}$
$^4I_{9/2}$	9438	0	4	$-6\sqrt{6}$	4	$4\sqrt{15}$	$-2\sqrt{10}$	$-12\sqrt{6}$	$4\sqrt{15}$	$8\sqrt{2}$	$-5\sqrt{5}$	$8\sqrt{2}$	$-6\sqrt{6}$	4	$12\sqrt{15}$	$-6\sqrt{6}$
$^2H_{9/2}(210x21)$	180180	$-20\sqrt{2}$	$30\sqrt{3}$	$-80\sqrt{2}$	$-5\sqrt{3}$	$54\sqrt{5}$	0	$-100\sqrt{2}$	$-9\sqrt{5}$	$2\sqrt{6}$	0	$23\sqrt{6}$	$-40\sqrt{2}$	$-25\sqrt{3}$	0	$120\sqrt{2}$
$^2H_{9/2}(210x11)$	198	-4	0	-4	$\sqrt{6}$	0	0	4	0	$\sqrt{3}$	0	$-2\sqrt{3}$	4	$-\sqrt{6}$	0	0
$^4G_{9/2}$	52272	0	$-5\sqrt{6}$	-65	$-5\sqrt{6}$	$-4\sqrt{10}$	$27\sqrt{15}$	13	$-4\sqrt{10}$	$2\sqrt{3}$	$9\sqrt{30}$	$2\sqrt{3}$	-65	$-5\sqrt{6}$	$-12\sqrt{10}$	-65
$^2G_{9/2}(210x21)$	16380	$-15\sqrt{2}$	$15\sqrt{3}$	$5\sqrt{2}$	$-25\sqrt{3}$	$12\sqrt{5}$	0	$-10\sqrt{2}$	$-17\sqrt{5}$	$-9\sqrt{6}$	0	$-\sqrt{6}$	$10\sqrt{2}$	$10\sqrt{3}$	0	$-15\sqrt{2}$
$^2G_{9/2}(210x20)$	4760	$-9\sqrt{2}$	$-5\sqrt{3}$	$-4\sqrt{2}$	$-8\sqrt{3}$	$-4\sqrt{5}$	0	$8\sqrt{2}$	$8\sqrt{5}$	$10\sqrt{6}$	0	$-2\sqrt{6}$	$-\sqrt{2}$	$13\sqrt{3}$	0	$5\sqrt{2}$
$^4F_{9/2}$	144	0	$-\sqrt{6}$	-1	$-\sqrt{6}$	0	$-\sqrt{15}$	1	0	$2\sqrt{3}$	$\sqrt{30}$	$2\sqrt{3}$	-1	$-\sqrt{6}$	0	-1
$^4G_{7/2}$	291060	0	$-55\sqrt{3}$	$70\sqrt{2}$	$-55\sqrt{3}$	$-44\sqrt{5}$	$27\sqrt{30}$	$-14\sqrt{2}$	$-44\sqrt{5}$	$11\sqrt{6}$	$18\sqrt{15}$	$11\sqrt{6}$	$70\sqrt{2}$	$-55\sqrt{3}$	$84\sqrt{5}$	$70\sqrt{2}$
$^2G_{7/2}(210x21)$	114660	105	$15\sqrt{6}$	-35	$-25\sqrt{6}$	$12\sqrt{10}$	0	70	$-17\sqrt{10}$	$-18\sqrt{3}$	0	$-2\sqrt{3}$	-70	$10\sqrt{6}$	0	105
$^2G_{7/2}(210x20)$	29106	63	$-5\sqrt{6}$	28	$-8\sqrt{6}$	$-4\sqrt{10}$	0	-56	$8\sqrt{10}$	$20\sqrt{3}$	0	$-4\sqrt{3}$	7	$13\sqrt{6}$	0	-35
$^4F_{7/2}$	252	0	$\sqrt{3}$	$2\sqrt{2}$	$\sqrt{3}$	0	$-\sqrt{30}$	$-2\sqrt{2}$	0	$-\sqrt{6}$	$2\sqrt{15}$	$-\sqrt{6}$	$2\sqrt{2}$	$\sqrt{3}$	0	$2\sqrt{2}$
$^2F_{7/2}(210x21)$	924	-7	$3\sqrt{6}$	5	$-5\sqrt{6}$	0	0	-2	$3\sqrt{10}$	$-2\sqrt{3}$	0	$-2\sqrt{3}$	-2	$2\sqrt{6}$	0	-3
$^2F_{7/2}(100x10)$	42	1	$\sqrt{6}$	0	0	0	0	0	0	0	0	0	1	$-\sqrt{6}$	0	-1
$^4D_{7/2}$	147	0	$\sqrt{10}$	0	$\sqrt{10}$	$-\sqrt{6}$	4	0	$-\sqrt{6}$	$\sqrt{5}$	$-\sqrt{2}$	$\sqrt{5}$	0	$\sqrt{10}$	0	0
$^4G_{5/2}$	25872	0	$5\sqrt{6}$	-35	$5\sqrt{6}$	$4\sqrt{10}$	$-3\sqrt{15}$	7	$4\sqrt{10}$	$-2\sqrt{3}$	$-\sqrt{30}$	$-2\sqrt{3}$	-35	$5\sqrt{6}$	$28\sqrt{10}$	-35
$^4F_{5/2}$	336	0	$-\sqrt{10}$	$\sqrt{15}$	$-\sqrt{10}$	0	3	$-\sqrt{15}$	0	$2\sqrt{5}$	$-3\sqrt{2}$	$2\sqrt{5}$	$\sqrt{15}$	$-\sqrt{10}$	0	$\sqrt{15}$
$^2F_{5/2}(210x21)$	924	$-7\sqrt{6}$	-3	$5\sqrt{6}$	5	0	0	$-2\sqrt{6}$	$-\sqrt{15}$	$\sqrt{2}$	0	$\sqrt{2}$	$-2\sqrt{6}$	-2	0	$-3\sqrt{6}$
$^2F_{5/2}(100x10)$	42	$\sqrt{6}$	-1	0	0	0	0	0	0	0	0	0	$\sqrt{6}$	1	0	$-\sqrt{6}$
$^4D_{5/2}$	882	0	$4\sqrt{5}$	0	$4\sqrt{5}$	$-4\sqrt{3}$	$-6\sqrt{2}$	0	$-4\sqrt{3}$	$2\sqrt{10}$	3	$2\sqrt{10}$	0	$4\sqrt{5}$	0	0
$^2D_{5/2}(210x21)$	42	0	$4\sqrt{5}$	0	$-2\sqrt{5}$	$-4\sqrt{3}$	0	0	$8\sqrt{3}$	$-\sqrt{10}$	0	$-4\sqrt{10}$	0	$-2\sqrt{5}$	0	0
$^2D_{5/2}(210x20)$	6468	0	$-18\sqrt{5}$	0	$9\sqrt{5}$	$18\sqrt{3}$	0	0	$-\sqrt{3}$	$-6\sqrt{10}$	0	$-3\sqrt{10}$	0	$9\sqrt{5}$	0	0

TABLE 7. $M = 5/2$ — *Continued*

	$\bar{3}1-\bar{1}$	$\bar{3}1-2$	$\bar{3}\bar{1}\bar{0}$	$\bar{3}\bar{1}-1$	$\bar{3}0\bar{0}$	$\bar{3}0-1$	$2\bar{2}-\bar{1}$	$2\bar{2}-2$	$21-\bar{1}$	$21-2$	$2\bar{1}\bar{0}$	$2\bar{1}-1$	$20\bar{0}$	$20-1$	$\bar{2}1\bar{0}$	$\bar{2}1-1$	$\bar{2}\bar{1}0$	$1\bar{1}0$
$^3L_{17/2}$	$-10\sqrt{6}$	$-3\sqrt{15}$	0	$5\sqrt{6}$	$-10\sqrt{6}$	$-5\sqrt{2}$	$9\sqrt{10}$	24	-15	0	$10\sqrt{3}$	45	-40	0	$-20\sqrt{3}$	-30	$10\sqrt{3}$	$5\sqrt{30}$
$^1L_{15/2}$	-110	$9\sqrt{10}$	0	55	-110	$10\sqrt{3}$	$33\sqrt{15}$	$-24\sqrt{6}$	$15\sqrt{6}$	0	$55\sqrt{2}$	$-45\sqrt{6}$	$40\sqrt{6}$	0	$-110\sqrt{2}$	$30\sqrt{6}$	$55\sqrt{2}$	$-30\sqrt{5}$
$^3K_{15/2}$	$2\sqrt{5}$	$-27\sqrt{2}$	0	$-17\sqrt{5}$	$18\sqrt{5}$	$-6\sqrt{15}$	$21\sqrt{3}$	$8\sqrt{30}$	$7\sqrt{30}$	0	$7\sqrt{10}$	$3\sqrt{30}$	$8\sqrt{30}$	0	$2\sqrt{10}$	$-10\sqrt{30}$	$-9\sqrt{10}$	10
$^3I_{15/2}$	$-4\sqrt{10}$	-15	$-\sqrt{30}$	$-4\sqrt{10}$	0	$-2\sqrt{30}$	0	0	$-3\sqrt{15}$	$-6\sqrt{6}$	$-2\sqrt{5}$	$-3\sqrt{15}$	0	$-6\sqrt{5}$	$-2\sqrt{5}$	$-3\sqrt{15}$	$-2\sqrt{5}$	0
$^1K_{13/2}$	$2\sqrt{10}$	27	0	$-17\sqrt{10}$	$18\sqrt{10}$	$3\sqrt{30}$	$21\sqrt{6}$	$-8\sqrt{15}$	$-7\sqrt{15}$	0	$14\sqrt{5}$	$-3\sqrt{15}$	$-8\sqrt{15}$	0	$4\sqrt{5}$	$10\sqrt{15}$	$-18\sqrt{5}$	$-5\sqrt{2}$
$^3I_{13/2}$	$-2\sqrt{10}$	0	$-\sqrt{30}$	$-2\sqrt{10}$	0	0	0	0	0	$3\sqrt{6}$	$-\sqrt{5}$	0	0	$3\sqrt{5}$	$-\sqrt{5}$	0	$-\sqrt{5}$	0
$^1I_{13/2}$	-10	$-3\sqrt{10}$	0	8	-6	$-\sqrt{3}$	0	$6\sqrt{6}$	0	0	$-7\sqrt{2}$	$-3\sqrt{6}$	$6\sqrt{6}$	0	$11\sqrt{2}$	$3\sqrt{6}$	$-4\sqrt{2}$	$-9\sqrt{5}$
$^1I_{11/2}$	$-12\sqrt{3}$	$13\sqrt{30}$	-135	$-12\sqrt{3}$	0	52	0	0	$39\sqrt{2}$	$-36\sqrt{5}$	$-3\sqrt{6}$	$39\sqrt{2}$	0	$-30\sqrt{6}$	$-3\sqrt{6}$	$39\sqrt{2}$	$-3\sqrt{6}$	0
$^3I_{11/2}$	$-15\sqrt{6}$	$4\sqrt{15}$	0	$12\sqrt{6}$	$-9\sqrt{6}$	$2\sqrt{2}$	0	-24	0	0	$-21\sqrt{3}$	12	-24	0	$33\sqrt{3}$	-12	$-12\sqrt{3}$	$6\sqrt{30}$
$^3H_{11/2}(210\times21)$	$-\sqrt{3}$	$-15\sqrt{30}$	0	$-24\sqrt{3}$	$-22\sqrt{3}$	-50	$-18\sqrt{5}$	$-20\sqrt{2}$	$2\sqrt{2}$	0	$-10\sqrt{6}$	$-27\sqrt{2}$	$-46\sqrt{2}$	0	$-\sqrt{6}$	$25\sqrt{2}$	$11\sqrt{6}$	$-6\sqrt{15}$
$^3H_{11/2}(210\times11)$	$\sqrt{6}$	0	0	0	$\sqrt{6}$	$2\sqrt{2}$	0	4	-4	0	$-\sqrt{3}$	0	-4	0	$-\sqrt{3}$	4	$2\sqrt{3}$	0
$^3G_{11/2}$	$-\sqrt{2}$	$4\sqrt{5}$	$-\sqrt{6}$	$-\sqrt{2}$	0	$-\sqrt{6}$	0	0	$-7\sqrt{3}$	$-\sqrt{30}$	-6	$-7\sqrt{3}$	0	-16	-6	$-7\sqrt{3}$	-6	0
$^1I_{9/2}$	$-12\sqrt{6}$	$4\sqrt{15}$	$30\sqrt{2}$	$-12\sqrt{6}$	0	$8\sqrt{2}$	0	0	12	$-3\sqrt{10}$	$-6\sqrt{3}$	12	0	$-5\sqrt{3}$	$-6\sqrt{3}$	12	$-6\sqrt{3}$	0
$^3H_{9/2}(210\times21)$	$4\sqrt{2}$	$-45\sqrt{5}$	0	$96\sqrt{2}$	$88\sqrt{2}$	$-25\sqrt{6}$	$24\sqrt{30}$	$-20\sqrt{3}$	$2\sqrt{3}$	0	80	$-27\sqrt{3}$	$-46\sqrt{3}$	0	8	$25\sqrt{3}$	-88	$-9\sqrt{10}$
$^3H_{9/2}(210\times11)$	-4	0	0	0	-4	$\sqrt{3}$	0	$\sqrt{6}$	$-\sqrt{6}$	0	$2\sqrt{2}$	0	$-\sqrt{6}$	0	$2\sqrt{2}$	$\sqrt{6}$	$-4\sqrt{2}$	0
$^3G_{9/2}$	13	$-4\sqrt{10}$	$24\sqrt{3}$	13	0	$2\sqrt{3}$	0	0	$7\sqrt{6}$	$-9\sqrt{15}$	$39\sqrt{2}$	$7\sqrt{6}$	0	$-72\sqrt{2}$	$39\sqrt{2}$	$7\sqrt{6}$	$39\sqrt{2}$	0
$^3G_{9/2}(210\times21)$	$7\sqrt{2}$	$5\sqrt{5}$	0	$3\sqrt{2}$	$-6\sqrt{2}$	$10\sqrt{6}$	$7\sqrt{30}$	$30\sqrt{3}$	$-9\sqrt{3}$	0	-20	$14\sqrt{3}$	$2\sqrt{3}$	0	14	$-5\sqrt{3}$	6	$-17\sqrt{10}$
$^3G_{9/2}(210\times20)$	$-7\sqrt{2}$	$-4\sqrt{5}$	0	$-\sqrt{2}$	$9\sqrt{2}$	$-8\sqrt{6}$	0	$-3\sqrt{3}$	$-11\sqrt{3}$	0	-12	$7\sqrt{3}$	$-3\sqrt{3}$	0	0	$4\sqrt{3}$	12	$-6\sqrt{10}$
$^3F_{9/2}$	1	0	0	1	0	$2\sqrt{3}$	0	0	$-\sqrt{6}$	$-\sqrt{15}$	$-\sqrt{2}$	$-\sqrt{6}$	0	0	$-\sqrt{2}$	$-\sqrt{6}$	$-\sqrt{2}$	0
$^1G_{7/2}$	$-14\sqrt{2}$	$-44\sqrt{5}$	$-84\sqrt{6}$	$-14\sqrt{2}$	0	$11\sqrt{6}$	0	0	$77\sqrt{3}$	$-9\sqrt{30}$	-84	$77\sqrt{3}$	0	-144	-84	$77\sqrt{3}$	-84	0
$^3G_{7/2}(210\times21)$	-49	$5\sqrt{10}$	0	-21	42	$20\sqrt{3}$	$-49\sqrt{15}$	$30\sqrt{6}$	$-9\sqrt{6}$	0	$70\sqrt{2}$	$14\sqrt{6}$	$2\sqrt{6}$	0	$-49\sqrt{2}$	$-5\sqrt{6}$	$-21\sqrt{2}$	$-34\sqrt{5}$
$^3G_{7/2}(210\times20)$	49	$-4\sqrt{10}$	0	7	-63	$-16\sqrt{3}$	0	$-3\sqrt{6}$	$-11\sqrt{6}$	0	$42\sqrt{2}$	$7\sqrt{6}$	$-3\sqrt{6}$	0	0	$4\sqrt{6}$	$-42\sqrt{2}$	$-12\sqrt{5}$
$^3F_{7/2}$	$-2\sqrt{2}$	0	0	$-2\sqrt{2}$	0	$-\sqrt{6}$	0	0	$\sqrt{3}$	$-\sqrt{30}$	4	$\sqrt{3}$	0	0	4	$\sqrt{3}$	4	0
$^3F_{7/2}(210\times21)$	-1	$-3\sqrt{10}$	0	3	2	$4\sqrt{3}$	$-\sqrt{15}$	$2\sqrt{6}$	$-\sqrt{6}$	0	$2\sqrt{2}$	$-2\sqrt{6}$	$2\sqrt{6}$	0	$-\sqrt{2}$	$3\sqrt{6}$	$-\sqrt{2}$	$6\sqrt{5}$
$^3F_{7/2}(100\times10)$	-1	0	0	1	1	0	0	$-\sqrt{6}$	$-\sqrt{6}$	0	0	$\sqrt{6}$	$\sqrt{6}$	0	0	0	0	0
$^3D_{7/2}$	0	$-\sqrt{6}$	0	0	0	$\sqrt{5}$	0	0	0	-6	0	0	0	$\sqrt{30}$	0	0	0	0
$^3G_{5/2}$	7	$4\sqrt{10}$	$-56\sqrt{3}$	7	0	$-2\sqrt{3}$	0	0	$-7\sqrt{6}$	$\sqrt{15}$	$21\sqrt{2}$	$-7\sqrt{6}$	0	$8\sqrt{2}$	$21\sqrt{2}$	$-7\sqrt{6}$	$21\sqrt{2}$	0
$^3F_{5/2}$	$-\sqrt{15}$	0	0	$-\sqrt{15}$	0	$2\sqrt{5}$	0	0	$-\sqrt{10}$	3	$\sqrt{30}$	$-\sqrt{10}$	0	0	$\sqrt{30}$	$-\sqrt{10}$	$\sqrt{30}$	0
$^3F_{5/2}(210\times21)$	$-\sqrt{6}$	$\sqrt{15}$	0	$3\sqrt{6}$	$2\sqrt{6}$	$-2\sqrt{2}$	$-3\sqrt{10}$	-2	1	0	$4\sqrt{3}$	2	-2	0	$-2\sqrt{3}$	-3	$-2\sqrt{3}$	$-\sqrt{30}$
$^3F_{5/2}(100\times10)$	$-\sqrt{6}$	0	0	$\sqrt{6}$	$\sqrt{6}$	0	0	1	1	0	0	-1	-1	0	0	0	0	0
$^3D_{5/2}$	0	$-4\sqrt{3}$	0	0	0	$2\sqrt{10}$	0	0	0	$9\sqrt{2}$	0	0	0	$-3\sqrt{15}$	0	0	0	0
$^3D_{5/2}(210\times21)$	0	$-4\sqrt{3}$	0	0	0	$5\sqrt{10}$	0	$-6\sqrt{5}$	$6\sqrt{5}$	0	0	0	$-6\sqrt{5}$	0	0	$-6\sqrt{5}$	0	$-6\sqrt{6}$
$^3D_{5/2}(210\times20)$	0	$-17\sqrt{3}$	0	0	0	$9\sqrt{10}$	0	$-8\sqrt{5}$	$-6\sqrt{5}$	0	0	$7\sqrt{5}$	$6\sqrt{5}$	0	0	$-\sqrt{5}$	0	$-\sqrt{6}$

TABLE 8. $M = 3/2$

	N^{-2}	$\bar{3}\bar{2}-\bar{3}$	$31-\bar{3}$	$3\bar{1}-\bar{2}$	$3\bar{1}-3$	$30-\bar{2}$	$30-3$	$\bar{3}\bar{0}-\bar{1}$	$\bar{3}\bar{0}-2$	$3-1-\bar{1}$	$3-1-2$	$\bar{3}2-\bar{3}$	$\bar{3}\bar{2}-\bar{2}$
$^2L_{17/2}$	19448	5	$-\sqrt{10}$	$3\sqrt{15}$	$3\sqrt{10}$	$-6\sqrt{5}$	0	$5\sqrt{2}$	$10\sqrt{5}$	$-10\sqrt{6}$	0	-6	0
$^2L_{15/2}$	136136	$5\sqrt{10}$	7	$15\sqrt{6}$	$-2\sqrt{1}$	$21\sqrt{2}$	0	$10\sqrt{5}$	$-35\sqrt{2}$	$14\sqrt{15}$	0	$-6\sqrt{10}$	0
$^2K_{15/2}$	40040	$3\sqrt{30}$	$9\sqrt{3}$	$27\sqrt{2}$	$5\sqrt{3}$	$19\sqrt{6}$	0	$6\sqrt{15}$	$-5\sqrt{6}$	$22\sqrt{5}$	0	$-2\sqrt{30}$	0
$^4I_{15/2}$	5005	$-\sqrt{15}$	$-4\sqrt{6}$	-15	$-4\sqrt{6}$	$-10\sqrt{3}$	$-9\sqrt{2}$	$-2\sqrt{30}$	$-10\sqrt{3}$	0	-10	$-\sqrt{15}$	$-2\sqrt{10}$
$^2K_{15/2}$	120120	$9\sqrt{15}$	$-9\sqrt{6}$	81	$-5\sqrt{6}$	$-38\sqrt{3}$	0	$9\sqrt{30}$	$10\sqrt{3}$	$-22\sqrt{10}$	0	$-6\sqrt{15}$	0
$^4I_{13/2}$	6435	$-2\sqrt{5}$	$2\sqrt{2}$	$-10\sqrt{3}$	$2\sqrt{2}$	5	$9\sqrt{6}$	$-4\sqrt{10}$	5	0	$10\sqrt{3}$	$-2\sqrt{5}$	$-3\sqrt{30}$
$^2I_{13/2}$	9009	$4\sqrt{10}$	-8	$5\sqrt{6}$	22	$-13\sqrt{2}$	0	$\sqrt{5}$	$20\sqrt{2}$	$-6\sqrt{15}$	0	$-5\sqrt{10}$	0
$^4I_{11/2}$	198198	$2\sqrt{2}$	$20\sqrt{5}$	$2\sqrt{30}$	$20\sqrt{5}$	$25\sqrt{10}$	$-27\sqrt{15}$	8	$25\sqrt{10}$	0	$-15\sqrt{30}$	$2\sqrt{2}$	$-72\sqrt{3}$
$^2I_{11/2}$	36036	32	$4\sqrt{10}$	$8\sqrt{15}$	$-11\sqrt{10}$	$13\sqrt{5}$	0	$4\sqrt{2}$	$-20\sqrt{5}$	$15\sqrt{6}$	0	-40	0
$^2H_{11/2}(210\times21)$	90090	$25\sqrt{2}$	$42\sqrt{5}$	$15\sqrt{30}$	$-7\sqrt{5}$	$28\sqrt{10}$	0	50	$7\sqrt{10}$	$-21\sqrt{3}$	0	$5\sqrt{2}$	0
$^2H_{11/2}(210\times11)$	396	4	0	0	$\sqrt{10}$	$\sqrt{5}$	0	$-2\sqrt{2}$	$-2\sqrt{5}$	$3\sqrt{6}$	0	-4	0
$^4G_{11/2}$	3630	$5\sqrt{3}$	$3\sqrt{30}$	$4\sqrt{5}$	$3\sqrt{30}$	$2\sqrt{15}$	$6\sqrt{10}$	$-\sqrt{6}$	$2\sqrt{15}$	0	$3\sqrt{5}$	$5\sqrt{3}$	$5\sqrt{2}$
$^4I_{9/2}$	198198	-28	$14\sqrt{10}$	$-28\sqrt{15}$	$14\sqrt{10}$	$35\sqrt{5}$	$-9\sqrt{30}$	$-56\sqrt{2}$	$35\sqrt{5}$	0	$-10\sqrt{15}$	-28	$42\sqrt{6}$
$^2H_{9/2}(210\times21)$	25740	-25	$12\sqrt{10}$	$-15\sqrt{15}$	$-2\sqrt{10}$	$16\sqrt{5}$	0	$-25\sqrt{2}$	$4\sqrt{5}$	$-6\sqrt{6}$	0	-5	0
$^2H_{9/2}(210\times11)$	1386	$-7\sqrt{2}$	0	0	$2\sqrt{5}$	$\sqrt{10}$	0	7	$-2\sqrt{10}$	$6\sqrt{3}$	0	$7\sqrt{2}$	0
$^4G_{9/2}$	121968	$-25\sqrt{6}$	$3\sqrt{15}$	$-20\sqrt{10}$	$3\sqrt{15}$	$\sqrt{30}$	$72\sqrt{5}$	$10\sqrt{3}$	$\sqrt{30}$	0	$18\sqrt{10}$	$-25\sqrt{6}$	-105
$^2G_{9/2}(210\times21)$	38220	$-10\sqrt{3}$	$9\sqrt{30}$	$-5\sqrt{5}$	$-14\sqrt{30}$	$2\sqrt{15}$	0	$-10\sqrt{6}$	$-12\sqrt{15}$	$-16\sqrt{2}$	0	$25\sqrt{3}$	0
$^2G_{9/2}(210\times20)$	9702	$-13\sqrt{3}$	$-3\sqrt{30}$	$4\sqrt{5}$	0	$4\sqrt{15}$	0	$8\sqrt{6}$	$4\sqrt{15}$	$24\sqrt{2}$	0	$8\sqrt{3}$	0
$^4F_{9/2}$	336	$-\sqrt{6}$	$-\sqrt{15}$	0	$-\sqrt{15}$	$\sqrt{30}$	0	$2\sqrt{3}$	$\sqrt{30}$	0	$2\sqrt{10}$	$-\sqrt{6}$	-1
$^4G_{7/2}$	97020	$5\sqrt{3}$	$-12\sqrt{30}$	$4\sqrt{5}$	$-12\sqrt{30}$	$-8\sqrt{15}$	$36\sqrt{10}$	$-\sqrt{6}$	$-8\sqrt{15}$	0	$18\sqrt{5}$	$5\sqrt{3}$	$105\sqrt{2}$
$^2G_{7/2}(210\times21)$	38220	$10\sqrt{6}$	$9\sqrt{15}$	$5\sqrt{10}$	$-14\sqrt{15}$	$\sqrt{30}$	0	$20\sqrt{3}$	$-6\sqrt{30}$	-16	0	$-25\sqrt{6}$	0
$^2G_{7/2}(210\times20)$	9702	$13\sqrt{6}$	$-3\sqrt{15}$	$-4\sqrt{10}$	0	$2\sqrt{30}$	0	$-16\sqrt{3}$	$2\sqrt{30}$	24	0	$-8\sqrt{6}$	0
$^4F_{7/2}$	84	$\sqrt{3}$	0	0	0	0	0	$-\sqrt{6}$	0	0	$2\sqrt{5}$	$\sqrt{3}$	$\sqrt{2}$
$^2F_{7/2}(210\times21)$	924	$-2\sqrt{2}$	$3\sqrt{5}$	$\sqrt{30}$	$-2\sqrt{5}$	$-\sqrt{10}$	0	-4	$2\sqrt{10}$	0	0	$5\sqrt{2}$	0
$^2F_{7/2}(100\times10)$	42	$\sqrt{2}$	$\sqrt{5}$	0	0	0	0	0	0	0	0	0	0
$^4D_{7/2}$	441	$\sqrt{10}$	4	$-\sqrt{6}$	4	$-\sqrt{2}$	$3\sqrt{3}$	$\sqrt{5}$	$-\sqrt{2}$	0	$-\sqrt{6}$	$\sqrt{10}$	0
$^4G_{5/2}$	25872	$-5\sqrt{30}$	$15\sqrt{3}$	$-20\sqrt{2}$	$15\sqrt{3}$	$5\sqrt{6}$	-24	$2\sqrt{15}$	$5\sqrt{6}$	0	$-6\sqrt{2}$	$-5\sqrt{30}$	$35\sqrt{5}$
$^4F_{5/2}$	1680	$\sqrt{10}$	-7	0	-7	$7\sqrt{2}$	0	$-2\sqrt{5}$	$7\sqrt{2}$	0	$-6\sqrt{6}$	$\sqrt{10}$	$3\sqrt{15}$
$^2F_{5/2}(210\times21)$	924	$-2\sqrt{5}$	$-3\sqrt{2}$	$5\sqrt{3}$	$2\sqrt{2}$	2	0	$-2\sqrt{10}$	-4	0	0	$5\sqrt{5}$	0
$^2F_{5/2}(100\times10)$	42	$\sqrt{5}$	$-\sqrt{2}$	0	0	0	0	0	0	0	0	0	0
$^4D_{5/2}$	4410	$8\sqrt{5}$	$2\sqrt{2}$	$-8\sqrt{3}$	$2\sqrt{2}$	-1	$-9\sqrt{6}$	$4\sqrt{10}$	-1	0	$6\sqrt{3}$	$8\sqrt{5}$	0
$^2D_{5/2}(210\times21)$	32340	$-9\sqrt{5}$	$-36\sqrt{2}$	$17\sqrt{3}$	$42\sqrt{2}$	48	0	$-9\sqrt{10}$	-36	$-6\sqrt{30}$	0	$-9\sqrt{5}$	0
$^2D_{5/2}(210\times20)$	4410	$\sqrt{10}$	8	$2\sqrt{6}$	14	$-17\sqrt{2}$	0	$-5\sqrt{5}$	$4\sqrt{2}$	$6\sqrt{15}$	0	$\sqrt{10}$	0
$^4F_{3/2}$	210	$-\sqrt{5}$	$\sqrt{2}$	0	$\sqrt{2}$	-2	0	$\sqrt{10}$	-2	0	$\sqrt{3}$	$-\sqrt{5}$	$\sqrt{30}$
$^4D_{3/2}$	630	$2\sqrt{10}$	-4	$-2\sqrt{6}$	-4	$\sqrt{2}$	$3\sqrt{3}$	$2\sqrt{5}$	$\sqrt{2}$	0	$-\sqrt{6}$	$2\sqrt{10}$	0
$^2D_{3/2}(210\times21)$	16170	$-9\sqrt{10}$	18	$17\sqrt{6}$	-21	$-12\sqrt{2}$	0	$-18\sqrt{5}$	$9\sqrt{2}$	$3\sqrt{15}$	0	$-9\sqrt{10}$	0
$^2D_{3/2}(210\times20)$	8820	$4\sqrt{5}$	$-4\sqrt{2}$	$8\sqrt{3}$	$-7\sqrt{2}$	17	0	$-10\sqrt{10}$	-4	$-3\sqrt{30}$	0	$4\sqrt{5}$	0
$^2P_{3/2}$	84	0	0	0	$\sqrt{6}$	$\sqrt{3}$	0	0	$-2\sqrt{3}$	$-\sqrt{10}$	0	0	0
$^4S_{3/2}$	7	0	0	0	0	0	-1	0	0	0	$\sqrt{2}$	0	0

TABLE 8. $M = 3/2$ — *Continued*

	$\overline{32}-3$	$\overline{31}-\overline{2}$	$\overline{31}-3$	$\overline{31}-\overline{1}$	$\overline{31}-2$	$\overline{30}-\overline{1}$	$\overline{30}-2$	$\overline{30}-1$	$22-\overline{2}$	$22-3$	$21-\overline{2}$	$21-3$	$21-\overline{1}$
$^2L_{17/2}$	1	$-5\sqrt{15}$	$-2\sqrt{10}$	0	$2\sqrt{15}$	$-20\sqrt{2}$	$-4\sqrt{5}$	$15\sqrt{2}$	24	$5\sqrt{6}$	$-3\sqrt{10}$	0	30
$^2L_{15/2}$	$\sqrt{10}$	$-25\sqrt{6}$	14	0	$10\sqrt{6}$	$-40\sqrt{5}$	$14\sqrt{2}$	$30\sqrt{5}$	$24\sqrt{10}$	$-7\sqrt{15}$	21	0	$30\sqrt{10}$
$^2K_{15/2}$	$-\sqrt{30}$	$-5\sqrt{2}$	$-14\sqrt{3}$	0	$-22\sqrt{2}$	$8\sqrt{15}$	$-14\sqrt{6}$	$-14\sqrt{15}$	$8\sqrt{30}$	$13\sqrt{5}$	$19\sqrt{3}$	0	$10\sqrt{30}$
$^4I_{15/2}$	$-\sqrt{15}$	-15	$-4\sqrt{6}$	$-4\sqrt{10}$	-15	$-2\sqrt{30}$	$-10\sqrt{3}$	$-2\sqrt{30}$	0	0	$-6\sqrt{6}$	-10	$-3\sqrt{15}$
$^2K_{13/2}$	$-3\sqrt{15}$	-15	$14\sqrt{6}$	0	-66	$12\sqrt{30}$	$28\sqrt{3}$	$-21\sqrt{30}$	$24\sqrt{15}$	$-13\sqrt{10}$	$-19\sqrt{6}$	0	$30\sqrt{15}$
$^4I_{13/2}$	$-2\sqrt{5}$	$-10\sqrt{3}$	$2\sqrt{2}$	$-6\sqrt{30}$	$-10\sqrt{3}$	$-4\sqrt{10}$	5	$-4\sqrt{10}$	0	0	$3\sqrt{2}$	$10\sqrt{3}$	$-6\sqrt{5}$
$^2I_{13/2}$	$\sqrt{10}$	$-10\sqrt{6}$	-14	0	$5\sqrt{6}$	$-8\sqrt{5}$	$-7\sqrt{2}$	$7\sqrt{5}$	$6\sqrt{10}$	$6\sqrt{15}$	-6	0	$-3\sqrt{10}$
$^4I_{11/2}$	$2\sqrt{2}$	$2\sqrt{30}$	$20\sqrt{5}$	$-144\sqrt{3}$	$2\sqrt{30}$	8	$25\sqrt{10}$	8	0	0	$30\sqrt{5}$	$-15\sqrt{30}$	$6\sqrt{2}$
$^2I_{11/2}$	8	$-16\sqrt{15}$	$7\sqrt{10}$	0	$8\sqrt{15}$	$-32\sqrt{2}$	$7\sqrt{5}$	$28\sqrt{2}$	48	$-15\sqrt{6}$	$3\sqrt{10}$	0	-24
$^2H_{11/2}(210\times21)$	$-30\sqrt{2}$	$3\sqrt{30}$	$-35\sqrt{5}$	0	$-18\sqrt{30}$	-46	$-35\sqrt{10}$	-4	$-20\sqrt{2}$	0	$28\sqrt{5}$	0	$-25\sqrt{2}$
$^2H_{11/2}(210\times11)$	0	0	$-\sqrt{10}$	0	0	$4\sqrt{2}$	$\sqrt{5}$	$-2\sqrt{2}$	4	$3\sqrt{6}$	$-\sqrt{10}$	0	-4
$^4G_{11/2}$	$5\sqrt{3}$	$4\sqrt{5}$	$3\sqrt{30}$	$-\sqrt{2}$	$4\sqrt{5}$	$-\sqrt{6}$	$2\sqrt{15}$	$-\sqrt{6}$	0	0	$-\sqrt{30}$	$3\sqrt{5}$	$-7\sqrt{3}$
$^4I_{9/2}$	-28	$-28\sqrt{15}$	$14\sqrt{10}$	$84\sqrt{6}$	$-28\sqrt{15}$	$-56\sqrt{2}$	$35\sqrt{5}$	$-56\sqrt{2}$	0	0	$21\sqrt{10}$	$-10\sqrt{15}$	-84
$^2H_{9/2}(210\times21)$	30	$-3\sqrt{15}$	$-10\sqrt{10}$	0	$18\sqrt{15}$	$23\sqrt{2}$	$-20\sqrt{5}$	$2\sqrt{2}$	20	0	$8\sqrt{10}$	0	25
$^2H_{9/2}(210\times11)$	0	0	$-2\sqrt{5}$	0	0	-14	$\sqrt{10}$	7	$-7\sqrt{2}$	$6\sqrt{3}$	$-2\sqrt{5}$	0	$7\sqrt{2}$
$^4G_{9/2}$	$-25\sqrt{6}$	$-20\sqrt{10}$	$3\sqrt{15}$	21	$-20\sqrt{10}$	$10\sqrt{3}$	$\sqrt{30}$	$10\sqrt{3}$	0	0	$-\sqrt{15}$	$18\sqrt{10}$	$35\sqrt{6}$
$^2G_{9/2}(210\times21)$	$-15\sqrt{3}$	$17\sqrt{5}$	$5\sqrt{30}$	0	$-12\sqrt{5}$	$\sqrt{6}$	$10\sqrt{15}$	$9\sqrt{6}$	$30\sqrt{3}$	$-5\sqrt{2}$	$\sqrt{30}$	0	$5\sqrt{3}$
$^2G_{9/2}(210\times20)$	$5\sqrt{3}$	$-8\sqrt{5}$	$3\sqrt{30}$	0	$4\sqrt{5}$	$2\sqrt{6}$	$-8\sqrt{15}$	$-10\sqrt{6}$	$-3\sqrt{3}$	$-24\sqrt{2}$	$-5\sqrt{30}$	0	$-4\sqrt{5}$
$^4F_{9/2}$	$-\sqrt{6}$	0	$-\sqrt{15}$	1	0	$2\sqrt{3}$	$\sqrt{30}$	$2\sqrt{3}$	0	0	$-\sqrt{15}$	$-2\sqrt{10}$	$-\sqrt{6}$
$^4G_{7/2}$	$5\sqrt{3}$	$4\sqrt{5}$	$-12\sqrt{30}$	$-21\sqrt{2}$	$4\sqrt{5}$	$-\sqrt{6}$	$-8\sqrt{15}$	$-\sqrt{6}$	0	0	$4\sqrt{30}$	$18\sqrt{5}$	$-7\sqrt{3}$
$^2G_{7/2}(210\times21)$	$15\sqrt{6}$	$-17\sqrt{10}$	$5\sqrt{15}$	0	$12\sqrt{10}$	$-2\sqrt{3}$	$5\sqrt{30}$	$-18\sqrt{3}$	$-30\sqrt{6}$	-5	$\sqrt{15}$	0	$-5\sqrt{6}$
$^2G_{7/2}(210\times20)$	$-5\sqrt{6}$	$8\sqrt{10}$	$3\sqrt{15}$	0	$-4\sqrt{10}$	$-4\sqrt{3}$	$-4\sqrt{30}$	$20\sqrt{3}$	$3\sqrt{6}$	-24	$-5\sqrt{15}$	0	$4\sqrt{6}$
$^4F_{7/2}$	$\sqrt{3}$	0	0	$-\sqrt{2}$	0	$-\sqrt{6}$	0	$-\sqrt{6}$	0	0	0	$-2\sqrt{5}$	$\sqrt{3}$
$^2F_{7/2}(210\times21)$	$-3\sqrt{2}$	$-\sqrt{30}$	$-\sqrt{5}$	0	0	2	$-\sqrt{10}$	2	$2\sqrt{2}$	$-5\sqrt{3}$	$-\sqrt{5}$	0	$-3\sqrt{2}$
$^2F_{7/2}(100\times10)$	$-\sqrt{2}$	0	$-\sqrt{5}$	0	0	0	0	0	$-\sqrt{2}$	0	$-\sqrt{5}$	0	0
$^4D_{7/2}$	$\sqrt{10}$	$-\sqrt{6}$	4	0	$-\sqrt{6}$	$\sqrt{5}$	$-\sqrt{2}$	$\sqrt{5}$	0	0	-6	$-\sqrt{6}$	0
$^4G_{5/2}$	$-5\sqrt{30}$	$-20\sqrt{2}$	$15\sqrt{3}$	$-7\sqrt{5}$	$-20\sqrt{2}$	$2\sqrt{15}$	$5\sqrt{6}$	$2\sqrt{15}$	0	0	$-5\sqrt{3}$	$-6\sqrt{2}$	$7\sqrt{30}$
$^4F_{5/2}$	$\sqrt{10}$	0	-7	$-3\sqrt{15}$	0	$-2\sqrt{5}$	$7\sqrt{2}$	$-2\sqrt{5}$	0	0	-7	$6\sqrt{6}$	$\sqrt{10}$
$^2F_{5/2}(210\times21)$	$-3\sqrt{5}$	$-5\sqrt{3}$	$\sqrt{2}$	0	0	$\sqrt{10}$	2	$\sqrt{10}$	$2\sqrt{5}$	$\sqrt{30}$	$\sqrt{2}$	0	$-3\sqrt{5}$
$^2F_{5/2}(100\times10)$	$-\sqrt{5}$	0	$\sqrt{2}$	0	0	0	0	0	$-\sqrt{5}$	0	$\sqrt{2}$	0	0
$^4D_{5/2}$	$8\sqrt{5}$	$-8\sqrt{3}$	$2\sqrt{2}$	0	$-8\sqrt{3}$	$4\sqrt{10}$	-1	$4\sqrt{10}$	0	0	$-3\sqrt{2}$	$6\sqrt{3}$	0
$^2D_{5/2}(210\times21)$	$18\sqrt{5}$	$\sqrt{3}$	$-6\sqrt{2}$	0	$-18\sqrt{3}$	$3\sqrt{10}$	-12	$6\sqrt{10}$	$-8\sqrt{5}$	$-8\sqrt{30}$	$24\sqrt{2}$	0	$\sqrt{5}$
$^2D_{5/2}(210\times20)$	$-2\sqrt{10}$	$-4\sqrt{6}$	-22	0	$2\sqrt{6}$	$4\sqrt{5}$	$13\sqrt{2}$	$\sqrt{5}$	$-3\sqrt{10}$	$-6\sqrt{15}$	18	0	$3\sqrt{10}$
$^4F_{3/2}$	$-\sqrt{5}$	0	$\sqrt{2}$	$-\sqrt{30}$	0	$\sqrt{10}$	-2	$\sqrt{10}$	0	0	$\sqrt{2}$	$-\sqrt{3}$	$-\sqrt{5}$
$^4D_{3/2}$	$2\sqrt{10}$	$-2\sqrt{6}$	-4	0	$-2\sqrt{6}$	$2\sqrt{5}$	$\sqrt{2}$	$2\sqrt{5}$	0	0	6	$-\sqrt{6}$	0
$^2D_{3/2}(210\times21)$	$18\sqrt{10}$	$\sqrt{6}$	3	0	$-18\sqrt{6}$	$6\sqrt{5}$	$3\sqrt{2}$	$12\sqrt{5}$	$-8\sqrt{10}$	$4\sqrt{15}$	-12	0	$\sqrt{10}$
$^2D_{3/2}(210\times20)$	$-8\sqrt{5}$	$-16\sqrt{3}$	$11\sqrt{2}$	0	$8\sqrt{3}$	$8\sqrt{10}$	-13	$+2\sqrt{10}$	$-12\sqrt{5}$	$3\sqrt{30}$	$-9\sqrt{2}$	0	$12\sqrt{5}$
$^2P_{3/2}$	0	0	$-\sqrt{6}$	0	0	0	$\sqrt{3}$	0	0	$-\sqrt{10}$	$-\sqrt{6}$	0	0
$^4S_{3/2}$	0	0	0	0	0	0	0	0	0	0	0	$\sqrt{2}$	0

TABLE 8. *M = 3/2 – Continued*

	$2\bar{1}-2$	$20-\bar{1}$	$20-2$	$\bar{2}0-1$	$\bar{2}1-\bar{1}$	$\bar{2}1-2$	$\bar{2}\bar{1}0$	$\bar{2}\bar{1}-1$	$\bar{2}00$	$\bar{2}0-1$	$1\bar{1}0$	$1\bar{1}-1$	$10\bar{1}$	$10-1$
$^2L_{17/2}$	$12\sqrt{10}$	$-20\sqrt{3}$	0	$30\sqrt{3}$	-45	$-9\sqrt{10}$	0	15	-40	$-10\sqrt{3}$	$5\sqrt{30}$	$15\sqrt{10}$	$-10\sqrt{10}$	0
$^2L_{15/2}$	-84	$14\sqrt{30}$	0	$-21\sqrt{30}$	$-45\sqrt{10}$	63	0	$15\sqrt{10}$	$-40\sqrt{30}$	$7\sqrt{30}$	$50\sqrt{3}$	-105	70	0
$^2K_{15/2}$	$20\sqrt{3}$	$22\sqrt{10}$	0	$-9\sqrt{10}$	$-3\sqrt{30}$	$-39\sqrt{3}$	0	$-7\sqrt{30}$	$8\sqrt{30}$	$-13\sqrt{10}$	10	$25\sqrt{3}$	$10\sqrt{3}$	0
$^4I_{15/2}$	$-6\sqrt{6}$	$-6\sqrt{5}$	$-16\sqrt{2}$	$-6\sqrt{5}$	$-3\sqrt{15}$	$-6\sqrt{6}$	$-2\sqrt{5}$	$-3\sqrt{15}$	0	$-6\sqrt{5}$	0	0	0	$-5\sqrt{2}$
$^2K_{13/2}$	$-20\sqrt{6}$	$-44\sqrt{5}$	0	$18\sqrt{5}$	$-9\sqrt{15}$	$39\sqrt{6}$	0	$-21\sqrt{15}$	$24\sqrt{15}$	$26\sqrt{5}$	$15\sqrt{2}$	$-25\sqrt{6}$	$-10\sqrt{6}$	0
$^4I_{13/2}$	$3\sqrt{2}$	$\sqrt{15}$	$16\sqrt{6}$	$\sqrt{15}$	$-6\sqrt{5}$	$3\sqrt{2}$	$-3\sqrt{15}$	$-6\sqrt{5}$	0	$\sqrt{15}$	0	0	0	$5\sqrt{6}$
$^2I_{13/2}$	18	$\sqrt{30}$	0	$-2\sqrt{30}$	$3\sqrt{10}$	-12	0	0	$6\sqrt{10}$	$\sqrt{30}$	$-15\sqrt{3}$	-30	30	0
$^4I_{11/2}$	$30\sqrt{5}$	$25\sqrt{6}$	$-48\sqrt{15}$	$25\sqrt{6}$	$6\sqrt{2}$	$30\sqrt{5}$	$-36\sqrt{6}$	$6\sqrt{2}$	0	$25\sqrt{6}$	0	0	0	$-15\sqrt{15}$
$^2I_{11/2}$	$-9\sqrt{10}$	$-5\sqrt{3}$	0	$10\sqrt{3}$	24	$6\sqrt{10}$	0	0	48	$-5\sqrt{3}$	$-12\sqrt{30}$	$15\sqrt{10}$	$-15\sqrt{10}$	0
$^2H_{11/2}(210\times21)$	$-28\sqrt{5}$	$-21\sqrt{6}$	0	$21\sqrt{6}$	$27\sqrt{2}$	0	0	$-2\sqrt{2}$	$-46\sqrt{2}$	0	$-6\sqrt{15}$	$-35\sqrt{5}$	$-14\sqrt{5}$	0
$^2H_{11/2}(210\times11)$	$\sqrt{10}$	$-3\sqrt{3}$	0	0	0	0	0	4	-4	$3\sqrt{3}$	0	$-\sqrt{10}$	$-\sqrt{10}$	0
$^4G_{11/2}$	$-\sqrt{30}$	-16	$-4\sqrt{10}$	-16	$-7\sqrt{3}$	$-\sqrt{30}$	-6	$-7\sqrt{3}$	0	-16	0	0	0	$-4\sqrt{10}$
$^4I_{9/2}$	$21\sqrt{10}$	$35\sqrt{3}$	$-16\sqrt{30}$	$35\sqrt{3}$	-84	$21\sqrt{10}$	$42\sqrt{3}$	-84	0	$35\sqrt{3}$	0	0	0	$-5\sqrt{30}$
$^2H_{9/2}(210\times21)$	$-8\sqrt{10}$	$-12\sqrt{3}$	0	$12\sqrt{3}$	-27	0	0	2	46	0	$3\sqrt{30}$	$-10\sqrt{10}$	$-4\sqrt{10}$	0
$^2H_{9/2}(210\times11)$	$2\sqrt{5}$	$-3\sqrt{6}$	0	0	0	0	0	$-7\sqrt{2}$	$7\sqrt{2}$	$3\sqrt{6}$	0	$-2\sqrt{5}$	$-2\sqrt{5}$	0
$^4G_{9/2}$	$-\sqrt{15}$	$-8\sqrt{2}$	$-48\sqrt{5}$	$-8\sqrt{2}$	$35\sqrt{6}$	$-\sqrt{15}$	$63\sqrt{2}$	$35\sqrt{6}$	0	$-8\sqrt{2}$	0	0	0	$-48\sqrt{5}$
$^2G_{9/2}(210\times21)$	$9\sqrt{30}$	-32	0	22	$-14\sqrt{3}$	$-10\sqrt{30}$	0	$9\sqrt{3}$	$2\sqrt{3}$	10	$-17\sqrt{10}$	$-5\sqrt{30}$	$12\sqrt{30}$	0
$^2G_{9/2}(210\times20)$	$4\sqrt{30}$	-8	0	16	$-7\sqrt{3}$	$\sqrt{30}$	0	$11\sqrt{3}$	$-3\sqrt{3}$	-8	$-6\sqrt{10}$	$-3\sqrt{30}$	$3\sqrt{30}$	0
$^4F_{9/2}$	$-\sqrt{15}$	0	0	0	$-\sqrt{6}$	$-\sqrt{15}$	$-\sqrt{2}$	$-\sqrt{6}$	0	0	0	0	0	0
$^4G_{7/2}$	$4\sqrt{30}$	64	$-24\sqrt{10}$	64	$-7\sqrt{3}$	$4\sqrt{30}$	-126	$-7\sqrt{3}$	0	64	0	0	0	$-24\sqrt{10}$
$^2G_{7/2}(210\times21)$	$9\sqrt{15}$	$-16\sqrt{2}$	0	$11\sqrt{2}$	$14\sqrt{6}$	$-10\sqrt{15}$	0	$-9\sqrt{6}$	$-2\sqrt{6}$	$5\sqrt{2}$	$34\sqrt{5}$	$-5\sqrt{15}$	$12\sqrt{15}$	0
$^2G_{7/2}(210\times20)$	$4\sqrt{15}$	$-4\sqrt{2}$	0	$8\sqrt{2}$	$7\sqrt{6}$	$\sqrt{15}$	0	$-11\sqrt{6}$	$3\sqrt{6}$	$-4\sqrt{2}$	$12\sqrt{5}$	$-3\sqrt{15}$	$3\sqrt{15}$	0
$^4F_{7/2}$	0	0	0	0	$\sqrt{3}$	0	2	$\sqrt{3}$	0	0	0	0	0	0
$^2F_{7/2}(210\times21)$	$3\sqrt{5}$	0	0	$-5\sqrt{6}$	$2\sqrt{2}$	$-2\sqrt{5}$	0	$\sqrt{2}$	$-2\sqrt{2}$	$5\sqrt{6}$	$2\sqrt{15}$	$\sqrt{5}$	$-4\sqrt{5}$	0
$^2F_{7/2}(100\times10)$	0	0	0	0	$-\sqrt{2}$	$\sqrt{5}$	0	$\sqrt{2}$	$\sqrt{2}$	0	0	$\sqrt{5}$	$\sqrt{5}$	0
$^4D_{7/2}$	-6	$\sqrt{30}$	$-2\sqrt{3}$	$\sqrt{30}$	0	-6	0	0	0	$\sqrt{30}$	0	0	0	$5\sqrt{3}$
$^4G_{5/2}$	$-5\sqrt{3}$	$-8\sqrt{10}$	16	$-8\sqrt{10}$	$7\sqrt{30}$	$-5\sqrt{3}$	$-21\sqrt{10}$	$7\sqrt{30}$	0	$-8\sqrt{10}$	0	0	0	16
$^4F_{5/2}$	-7	0	0	0	$\sqrt{10}$	-7	$3\sqrt{30}$	$\sqrt{10}$	0	0	0	0	0	0
$^2F_{5/2}(210\times21)$	$-3\sqrt{2}$	0	0	$2\sqrt{15}$	$2\sqrt{5}$	$2\sqrt{2}$	0	$\sqrt{5}$	$2\sqrt{5}$	$-2\sqrt{15}$	$5\sqrt{6}$	$-\sqrt{2}$	$4\sqrt{2}$	0
$^2F_{5/2}(100\times10)$	0	0	0	0	$-\sqrt{5}$	$-\sqrt{2}$	0	$\sqrt{5}$	$\sqrt{5}$	0	0	$-\sqrt{2}$	$-\sqrt{2}$	0
$^4D_{5/2}$	$-3\sqrt{2}$	$\sqrt{15}$	$6\sqrt{6}$	$\sqrt{15}$	0	$-3\sqrt{2}$	0	0	0	$\sqrt{15}$	0	0	0	$-15\sqrt{6}$
$^2D_{5/2}(210\times21)$	$-8\sqrt{2}$	$-12\sqrt{15}$	0	$-4\sqrt{15}$	$-7\sqrt{5}$	$-16\sqrt{2}$	0	$6\sqrt{5}$	$6\sqrt{5}$	$16\sqrt{15}$	$-\sqrt{6}$	$34\sqrt{2}$	$36\sqrt{2}$	0
$^2D_{5/2}(210\times20)$	-6	$-\sqrt{30}$	0	$2\sqrt{30}$	0	-12	0	$-3\sqrt{10}$	$-3\sqrt{10}$	$-\sqrt{30}$	$-6\sqrt{3}$	-6	6	0
$^4F_{3/2}$	$\sqrt{2}$	0	0	0	$-\sqrt{5}$	$\sqrt{2}$	$2\sqrt{15}$	$-\sqrt{5}$	0	0	0	0	0	0
$^4D_{3/2}$	6	$-\sqrt{30}$	$-2\sqrt{3}$	$-\sqrt{30}$	0	6	0	0	0	$-\sqrt{30}$	0	0	0	$5\sqrt{3}$
$^2D_{3/2}(210\times21)$	4	$3\sqrt{30}$	0	$\sqrt{30}$	$-7\sqrt{10}$	8	0	$6\sqrt{10}$	$6\sqrt{10}$	$-4\sqrt{30}$	$-2\sqrt{3}$	-17	-18	0
$^2D_{3/2}(210\times20)$	$3\sqrt{2}$	$\sqrt{15}$	0	$-2\sqrt{15}$	0	$6\sqrt{2}$	0	$-12\sqrt{5}$	$-12\sqrt{5}$	$\sqrt{15}$	$-12\sqrt{6}$	$3\sqrt{2}$	$-3\sqrt{2}$	0
$^2P_{3/2}$	$\sqrt{6}$	$\sqrt{5}$	0	0	0	0	0	0	0	$-\sqrt{5}$	0	$-\sqrt{6}$	$-\sqrt{6}$	0
$^4S_{3/2}$	0	0	-1	0	0	0	0	0	0	0	0	0	0	1

TABLE 9. $M = \frac{1}{2}$

	N^{-2}	$3\bar{1}-\bar{3}$	$30-\bar{3}$	$\bar{3}0-\bar{2}$	$30-3$	$3-1-\bar{2}$	$3-1-3$	$3-\bar{1}-2$	$\bar{3}2-\bar{3}$	$\bar{3}1-\bar{3}$	$\bar{3}1-\bar{2}$	$\bar{3}\bar{1}-3$	$\bar{3}0-\bar{2}$	$\bar{3}0-3$
$^2L_{17/2}$	4862	$2\sqrt{2}$	$-\sqrt{6}$	4	$2\sqrt{6}$	$-4\sqrt{3}$	0	$5\sqrt{3}$	0	$-3\sqrt{2}$	0	$\sqrt{2}$	-10	$-\sqrt{6}$
$^2L_{15/2}$	58344	$6\sqrt{3}$	8	$6\sqrt{6}$	-16	$16\sqrt{2}$	0	$-20\sqrt{2}$	0	$-9\sqrt{3}$	0	$3\sqrt{3}$	$-15\sqrt{6}$	8
$^2K_{15/2}$	51480	$14\sqrt{3}$	24	$14\sqrt{6}$	0	$32\sqrt{2}$	0	$-20\sqrt{2}$	0	$-5\sqrt{3}$	0	$-9\sqrt{3}$	$5\sqrt{6}$	-24
$^4I_{15/2}$	6435	$-4\sqrt{6}$	$-9\sqrt{2}$	$-10\sqrt{3}$	$-9\sqrt{2}$	-10	$-4\sqrt{6}$	-10	$-\sqrt{15}$	$-4\sqrt{6}$	-15	$-4\sqrt{6}$	$-10\sqrt{3}$	$-9\sqrt{2}$
$^2K_{13/2}$	90090	$14\sqrt{6}$	$-21\sqrt{2}$	$28\sqrt{3}$	0	-56	0	35	0	$-5\sqrt{6}$	0	$-9\sqrt{6}$	$10\sqrt{3}$	$21\sqrt{2}$
$^4I_{13/2}$	25740	$-6\sqrt{6}$	$9\sqrt{2}$	$-15\sqrt{3}$	$9\sqrt{2}$	10	$14\sqrt{6}$	10	$-4\sqrt{15}$	$-6\sqrt{6}$	-60	$-6\sqrt{6}$	$-15\sqrt{3}$	$9\sqrt{2}$
$^2I_{13/2}$	36036	$14\sqrt{3}$	-21	$7\sqrt{6}$	42	$-28\sqrt{2}$	0	$35\sqrt{2}$	0	$-22\sqrt{3}$	0	$8\sqrt{3}$	$-20\sqrt{6}$	-21
$^4I_{11/2}$	28314	8	$9\sqrt{3}$	$10\sqrt{2}$	$9\sqrt{3}$	$5\sqrt{6}$	-36	$5\sqrt{6}$	$-6\sqrt{10}$	8	$-30\sqrt{6}$	8	$10\sqrt{2}$	$9\sqrt{3}$
$^2I_{11/2}$	5148	$7\sqrt{2}$	$3\sqrt{6}$	7	$-6\sqrt{6}$	$8\sqrt{3}$	0	$-10\sqrt{3}$	0	$-11\sqrt{2}$	0	$4\sqrt{2}$	-20	$3\sqrt{6}$
$^2H_{11/2}(210\times21)$	2574	$5\sqrt{5}$	$4\sqrt{15}$	$5\sqrt{10}$	0	$\sqrt{30}$	0	$\sqrt{30}$	0	$\sqrt{5}$	0	$-6\sqrt{5}$	$-\sqrt{10}$	$-4\sqrt{15}$
$^2H_{11/2}(210\times11)$	2772	$5\sqrt{2}$	$\sqrt{6}$	-5	0	$8\sqrt{3}$	0	$-10\sqrt{3}$	0	$-5\sqrt{2}$	0	0	10	$-\sqrt{6}$
$^4G_{11/2}$	5082	$3\sqrt{30}$	$6\sqrt{10}$	$2\sqrt{15}$	$6\sqrt{10}$	$3\sqrt{5}$	$3\sqrt{30}$	$3\sqrt{5}$	$5\sqrt{3}$	$3\sqrt{30}$	$4\sqrt{5}$	$3\sqrt{30}$	$2\sqrt{15}$	$6\sqrt{10}$
$^4I_{9/2}$	132132	$-14\sqrt{10}$	$9\sqrt{30}$	$-35\sqrt{5}$	$9\sqrt{30}$	$10\sqrt{15}$	$-14\sqrt{10}$	$10\sqrt{15}$	28	$-14\sqrt{10}$	$28\sqrt{15}$	$-14\sqrt{10}$	$-35\sqrt{5}$	$9\sqrt{30}$
$^2H_{9/2}(210\times21)$	12870	$-5\sqrt{30}$	$10\sqrt{10}$	$-10\sqrt{15}$	0	$5\sqrt{5}$	0	$5\sqrt{5}$	0	$-\sqrt{30}$	0	$6\sqrt{30}$	$2\sqrt{15}$	$-10\sqrt{10}$
$^2H_{9/2}(210\times11)$	2772	$-2\sqrt{15}$	$\sqrt{5}$	$\sqrt{30}$	0	$4\sqrt{10}$	0	$-5\sqrt{10}$	0	$2\sqrt{15}$	0	0	$-2\sqrt{30}$	$-\sqrt{5}$
$^4G_{9/2}$	182952	$-21\sqrt{15}$	$24\sqrt{5}$	$-7\sqrt{30}$	$24\sqrt{5}$	$6\sqrt{10}$	$45\sqrt{15}$	$6\sqrt{10}$	$-45\sqrt{6}$	$-21\sqrt{15}$	$-36\sqrt{10}$	$-21\sqrt{15}$	$-7\sqrt{30}$	$24\sqrt{5}$
$^2G_{9/2}(210\times21)$	57330	$-5\sqrt{30}$	$15\sqrt{10}$	$-10\sqrt{15}$	$-30\sqrt{10}$	$-5\sqrt{5}$	0	$-10\sqrt{5}$	0	$14\sqrt{30}$	0	$-9\sqrt{30}$	$12\sqrt{15}$	$15\sqrt{10}$
$^2G_{9/2}(210\times20)$	29106	$-6\sqrt{15}$	$-3\sqrt{5}$	$8\sqrt{30}$	$6\sqrt{5}$	$18\sqrt{10}$	0	$-6\sqrt{10}$	0	0	0	$6\sqrt{15}$	$-4\sqrt{30}$	$-3\sqrt{5}$
$^4F_{9/2}$	504	$-\sqrt{15}$	0	$\sqrt{30}$	0	$2\sqrt{10}$	$\sqrt{15}$	$2\sqrt{10}$	$-\sqrt{6}$	$-\sqrt{15}$	0	$-\sqrt{15}$	$\sqrt{30}$	0
$^4G_{7/2}$	58212	$-3\sqrt{30}$	$-12\sqrt{10}$	$-2\sqrt{15}$	$-12\sqrt{10}$	$-6\sqrt{5}$	$18\sqrt{30}$	$-6\sqrt{5}$	$45\sqrt{3}$	$-3\sqrt{30}$	$36\sqrt{5}$	$-3\sqrt{30}$	$-2\sqrt{15}$	$-12\sqrt{10}$
$^2G_{7/2}(210\times21)$	22932	$5\sqrt{15}$	$12\sqrt{5}$	$5\sqrt{30}$	$-24\sqrt{5}$	$-2\sqrt{10}$	0	$-4\sqrt{10}$	0	$-14\sqrt{15}$	0	$9\sqrt{15}$	$-6\sqrt{30}$	$12\sqrt{5}$
$^2G_{7/2}(210\times20)$	29106	$15\sqrt{3}$	-6	$-20\sqrt{6}$	12	$36\sqrt{2}$	0	$-12\sqrt{2}$	0	0	0	$-15\sqrt{3}$	$10\sqrt{6}$	-6
$^4F_{7/2}$	252	$\sqrt{6}$	0	$-2\sqrt{3}$	0	2	$2\sqrt{6}$	2	$\sqrt{15}$	$\sqrt{6}$	0	$\sqrt{6}$	$-2\sqrt{3}$	0
$^2F_{7/2}(210\times21)$	924	$\sqrt{3}$	4	$\sqrt{6}$	0	$-2\sqrt{2}$	0	$4\sqrt{2}$	0	$2\sqrt{3}$	0	$-3\sqrt{3}$	$-2\sqrt{6}$	-4
$^2F_{7/2}(100\times10)$	42	$\sqrt{3}$	2	0	0	0	0	0	0	0	0	$-\sqrt{3}$	0	-2
$^4D_{7/2}$	735	4	$3\sqrt{3}$	$-\sqrt{2}$	$3\sqrt{3}$	$-\sqrt{6}$	4	$-\sqrt{6}$	$\sqrt{10}$	4	$-\sqrt{6}$	4	$-\sqrt{2}$	$3\sqrt{3}$
$^4G_{5/2}$	12936	$-15\sqrt{3}$	24	$-5\sqrt{6}$	24	$6\sqrt{2}$	$-15\sqrt{3}$	$6\sqrt{2}$	$5\sqrt{30}$	$-15\sqrt{3}$	$20\sqrt{2}$	$-15\sqrt{3}$	$-5\sqrt{6}$	24
$^4F_{5/2}$	840	-1	0	$\sqrt{2}$	0	$2\sqrt{6}$	-9	$2\sqrt{6}$	$3\sqrt{10}$	-1	0	-1	$\sqrt{2}$	0
$^2F_{5/2}(210\times21)$	462	$\sqrt{2}$	$-\sqrt{6}$	2	0	$\sqrt{3}$	0	$-2\sqrt{3}$	0	$2\sqrt{2}$	0	$-3\sqrt{2}$	-4	$\sqrt{6}$
$^2F_{5/2}(100\times10)$	42	2	$-\sqrt{3}$	0	0	0	0	0	0	0	0	-2	0	$\sqrt{3}$
$^2D_{5/2}$	8820	$10\sqrt{2}$	$-3\sqrt{6}$	-5	$-3\sqrt{6}$	$2\sqrt{3}$	$-18\sqrt{2}$	$2\sqrt{3}$	$12\sqrt{5}$	$10\sqrt{2}$	$-12\sqrt{3}$	$10\sqrt{2}$	-5	$-3\sqrt{6}$
$^2D_{5/2}(210\times21)$	16170	$3\sqrt{2}$	$-6\sqrt{6}$	6	$12\sqrt{6}$	$9\sqrt{3}$	0	$-3\sqrt{3}$	0	$-21\sqrt{2}$	0	$18\sqrt{2}$	18	$-6\sqrt{6}$
$^2D_{5/2}(210\times20)$	8820	22	$-9\sqrt{3}$	$-13\sqrt{2}$	$18\sqrt{3}$	$-2\sqrt{6}$	0	$-11\sqrt{6}$	0	-14	0	-8	$-4\sqrt{2}$	$-9\sqrt{3}$
$^4F_{3/2}$	70	$-\sqrt{2}$	0	2	0	$-\sqrt{3}$	$\sqrt{2}$	$-\sqrt{3}$	$\sqrt{5}$	$-\sqrt{2}$	0	$-\sqrt{2}$	2	0
$^4D_{3/2}$	630	0	-3	0	-3	$\sqrt{2}$	$4\sqrt{3}$	$\sqrt{2}$	$2\sqrt{30}$	0	$-6\sqrt{2}$	0	0	-3
$^2D_{3/2}(210\times21)$	5390	3	$4\sqrt{3}$	$3\sqrt{2}$	$-8\sqrt{3}$	$-3\sqrt{6}$	0	$\sqrt{6}$	0	-21	0	18	$9\sqrt{2}$	$4\sqrt{3}$
$^2D_{3/2}(210\times20)$	8820	$11\sqrt{6}$	$9\sqrt{2}$	$-13\sqrt{3}$	$-18\sqrt{2}$	4	0	22	0	$-7\sqrt{6}$	0	$-4\sqrt{6}$	$-4\sqrt{3}$	$9\sqrt{2}$
$^4P_{3/2}$	252	$\sqrt{6}$	$3\sqrt{2}$	$-\sqrt{3}$	0	-4	0	-2	0	$-\sqrt{6}$	0	0	$2\sqrt{3}$	$-3\sqrt{2}$
$^4S_{3/2}$	21	0	-1	0	-1	$\sqrt{2}$	0	$\sqrt{2}$	0	0	0	0	0	-1
$^4D_{1/2}$	1260	$-2\sqrt{6}$	$3\sqrt{2}$	$\sqrt{3}$	$3\sqrt{2}$	-2	$-2\sqrt{6}$	-2	$4\sqrt{15}$	$-2\sqrt{6}$	-12	$-2\sqrt{6}$	$\sqrt{3}$	$3\sqrt{2}$
$^4P_{1/2}$	252	$2\sqrt{3}$	-3	$-\sqrt{6}$	0	$2\sqrt{2}$	0	$\sqrt{2}$	0	$-2\sqrt{3}$	0	0	$2\sqrt{6}$	3

TABLE 9. $M = \frac{1}{2}$ – Continued

	$\bar{3}\bar{0}-\bar{1}$	$\bar{3}\bar{0}-2$	$\bar{3}-1-\bar{1}$	$\bar{3}-1-2$	$2\bar{2}-\bar{3}$	$21-\bar{3}$	$2\bar{1}-\bar{2}$	$2\bar{1}-3$	$20-\bar{2}$	$20-3$	$\bar{2}\bar{0}-\bar{1}$	$\bar{2}\bar{0}-2$	$2-1-\bar{1}$	$2-1-2$
$^1L_{17/2}$	0	6	$-2\sqrt{30}$	$-\sqrt{3}$	$\sqrt{30}$	$-\sqrt{3}$	$9\sqrt{2}$	$5\sqrt{3}$	$-4\sqrt{6}$	0	$2\sqrt{15}$	$8\sqrt{6}$	$-9\sqrt{5}$	0
$^1L_{15/2}$	0	$9\sqrt{6}$	$-18\sqrt{5}$	$4\sqrt{2}$	$9\sqrt{5}$	$4\sqrt{2}$	$27\sqrt{3}$	$-20\sqrt{2}$	32	0	$9\sqrt{10}$	-64	$12\sqrt{30}$	0
$^1K_{15/2}$	0	$-19\sqrt{6}$	$22\sqrt{5}$	$-12\sqrt{2}$	$13\sqrt{5}$	$12\sqrt{2}$	$39\sqrt{3}$	$20\sqrt{2}$	64	0	$13\sqrt{10}$	0	$12\sqrt{30}$	0
$^1I_{13/2}$	$-2\sqrt{30}$	$-10\sqrt{3}$	0	-10	0	-10	$-6\sqrt{6}$	-10	$-16\sqrt{2}$	$-10\sqrt{3}$	$-6\sqrt{5}$	$-16\sqrt{2}$	0	$-6\sqrt{6}$
$^3K_{13/2}$	0	$-38\sqrt{3}$	$22\sqrt{10}$	21	$13\sqrt{10}$	-21	$39\sqrt{6}$	-35	$-56\sqrt{2}$	0	$26\sqrt{5}$	0	$-21\sqrt{15}$	0
$^1I_{13/2}$	$-8\sqrt{30}$	$-15\sqrt{3}$	0	10	0	10	$-9\sqrt{6}$	10	$16\sqrt{2}$	$35\sqrt{3}$	$-9\sqrt{5}$	$16\sqrt{2}$	0	$21\sqrt{6}$
$^3I_{13/2}$	0	$13\sqrt{6}$	$-18\sqrt{5}$	$-7\sqrt{2}$	$18\sqrt{5}$	$-7\sqrt{2}$	$12\sqrt{3}$	$35\sqrt{2}$	-14	0	$-3\sqrt{10}$	28	0	0
$^1I_{11/2}$	$-24\sqrt{5}$	$10\sqrt{2}$	0	$5\sqrt{6}$	0	$5\sqrt{6}$	12	$5\sqrt{6}$	$16\sqrt{3}$	$-45\sqrt{2}$	$2\sqrt{30}$	$16\sqrt{3}$	0	-54
$^3I_{11/2}$	0	13	$-3\sqrt{30}$	$2\sqrt{3}$	$3\sqrt{30}$	$2\sqrt{3}$	$6\sqrt{2}$	$-10\sqrt{3}$	$2\sqrt{6}$	0	$-\sqrt{15}$	$-4\sqrt{6}$	0	0
$^3H_{11/2}(210\times21)$	0	$-4\sqrt{10}$	$-3\sqrt{3}$	$-2\sqrt{30}$	0	$2\sqrt{30}$	0	$-\sqrt{30}$	$2\sqrt{15}$	0	0	0	$-9\sqrt{2}$	0
$^3H_{11/2}(210\times11)$	0	-5	$3\sqrt{30}$	$2\sqrt{3}$	$3\sqrt{30}$	$-2\sqrt{3}$	0	$10\sqrt{3}$	$-4\sqrt{6}$	0	$-3\sqrt{15}$	0	0	0
$^3G_{11/2}$	$-\sqrt{6}$	$2\sqrt{15}$	0	$3\sqrt{5}$	0	$3\sqrt{5}$	$-\sqrt{30}$	$3\sqrt{5}$	$-4\sqrt{10}$	$2\sqrt{15}$	-16	$-4\sqrt{10}$	0	$-\sqrt{30}$
$^1I_{9/2}$	$56\sqrt{2}$	$-35\sqrt{5}$	0	$10\sqrt{15}$	0	$10\sqrt{15}$	$-21\sqrt{10}$	$10\sqrt{15}$	$16\sqrt{30}$	$-35\sqrt{5}$	$-35\sqrt{3}$	$16\sqrt{30}$	0	$-21\sqrt{10}$
$^3H_{9/2}(210\times21)$	0	$8\sqrt{15}$	$9\sqrt{2}$	$-10\sqrt{5}$	0	$10\sqrt{5}$	0	$-5\sqrt{5}$	$5\sqrt{10}$	0	0	0	$-15\sqrt{3}$	0
$^3H_{9/2}(210\times11)$	0	$\sqrt{30}$	-18	$\sqrt{10}$	-18	$-\sqrt{10}$	0	$5\sqrt{10}$	$-4\sqrt{5}$	0	$9\sqrt{2}$	0	0	0
$^3G_{9/2}$	$18\sqrt{3}$	$-7\sqrt{30}$	0	$6\sqrt{10}$	0	$6\sqrt{10}$	$7\sqrt{15}$	$6\sqrt{10}$	$-16\sqrt{5}$	$15\sqrt{30}$	$56\sqrt{2}$	$-16\sqrt{5}$	0	$-15\sqrt{15}$
$^3G_{9/2}(210\times21)$	0	$-2\sqrt{15}$	$-16\sqrt{2}$	$15\sqrt{5}$	$-5\sqrt{2}$	$15\sqrt{5}$	$10\sqrt{30}$	$-10\sqrt{5}$	$-5\sqrt{10}$	0	-10	$10\sqrt{10}$	$-35\sqrt{3}$	0
$^3G_{9/2}(210\times20)$	0	$-4\sqrt{30}$	48	$-12\sqrt{10}$	-48	$-12\sqrt{10}$	$-2\sqrt{15}$	$-6\sqrt{10}$	$-13\sqrt{5}$	0	$8\sqrt{2}$	$26\sqrt{5}$	0	0
$^1F_{9/2}$	$2\sqrt{3}$	$\sqrt{30}$	0	$2\sqrt{10}$	0	$-2\sqrt{10}$	$-\sqrt{15}$	$-2\sqrt{10}$	0	$-\sqrt{30}$	0	0	0	$\sqrt{15}$
$^3G_{7/2}$	$-9\sqrt{6}$	$-2\sqrt{15}$	0	$-6\sqrt{5}$	0	$-6\sqrt{5}$	$\sqrt{30}$	$-6\sqrt{5}$	$8\sqrt{10}$	$12\sqrt{15}$	16	$8\sqrt{10}$	0	$-6\sqrt{30}$
$^3G_{7/2}(210\times21)$	0	$\sqrt{30}$	16	$6\sqrt{10}$	5	$6\sqrt{10}$	$-10\sqrt{15}$	$-4\sqrt{10}$	$-4\sqrt{5}$	0	$5\sqrt{2}$	$8\sqrt{5}$	$-14\sqrt{6}$	0
$^3G_{7/2}(210\times20)$	0	$10\sqrt{6}$	$-24\sqrt{5}$	$-24\sqrt{2}$	$24\sqrt{5}$	$-24\sqrt{2}$	$5\sqrt{3}$	$-12\sqrt{2}$	-26	0	$-4\sqrt{10}$	52	0	0
$^1F_{7/2}$	$-\sqrt{30}$	$-2\sqrt{3}$	0	2	0	-2	$\sqrt{6}$	-2	0	$-4\sqrt{3}$	0	0	0	$2\sqrt{6}$
$^3F_{7/2}(210\times21)$	0	$\sqrt{6}$	0	$-2\sqrt{2}$	$-3\sqrt{5}$	$2\sqrt{2}$	$2\sqrt{3}$	$-4\sqrt{2}$	-4	0	$-3\sqrt{10}$	0	$2\sqrt{30}$	0
$^3F_{7/2}(100\times10)$	0	0	0	0	0	0	$-\sqrt{3}$	0	-2	0	0	0	0	0
$^1D_{7/2}$	$\sqrt{5}$	$-\sqrt{2}$	0	$-\sqrt{6}$	0	$-\sqrt{6}$	-6	$-\sqrt{6}$	$-2\sqrt{3}$	$-\sqrt{2}$	$\sqrt{30}$	$-2\sqrt{3}$	0	-6
$^3G_{5/2}$	$-2\sqrt{15}$	$-5\sqrt{6}$	0	$6\sqrt{2}$	0	$6\sqrt{2}$	$5\sqrt{3}$	$6\sqrt{2}$	-16	$-5\sqrt{6}$	$8\sqrt{10}$	-16	0	$5\sqrt{3}$
$^1F_{5/2}$	$-6\sqrt{5}$	$\sqrt{2}$	0	$2\sqrt{6}$	0	$-2\sqrt{6}$	-1	$-2\sqrt{6}$	0	$9\sqrt{2}$	0	0	0	-9
$^3F_{5/2}(210\times21)$	0	2	0	$\sqrt{3}$	$-\sqrt{30}$	$-\sqrt{3}$	$2\sqrt{2}$	$2\sqrt{3}$	$\sqrt{6}$	0	$-2\sqrt{15}$	0	$-3\sqrt{5}$	0
$^3F_{5/2}(100\times10)$	0	0	0	0	0	0	-2	0	$\sqrt{3}$	0	0	0	0	0
$^1D_{5/2}$	$6\sqrt{10}$	-5	0	$2\sqrt{3}$	0	$2\sqrt{3}$	$-15\sqrt{2}$	$2\sqrt{3}$	$2\sqrt{6}$	9	$5\sqrt{15}$	$2\sqrt{6}$	0	$27\sqrt{2}$
$^3D_{5/2}(210\times21)$	0	-24	$-3\sqrt{30}$	$-6\sqrt{3}$	$-4\sqrt{30}$	$-6\sqrt{3}$	$8\sqrt{2}$	$-3\sqrt{3}$	$9\sqrt{6}$	0	$-8\sqrt{15}$	$-18\sqrt{6}$	$-21\sqrt{5}$	0
$^3D_{5/2}(210\times20)$	0	$17\sqrt{2}$	$6\sqrt{15}$	$13\sqrt{6}$	$-6\sqrt{15}$	$13\sqrt{6}$	12	$-11\sqrt{6}$	$-4\sqrt{3}$	0	$\sqrt{30}$	$8\sqrt{3}$	0	0
$^3F_{3/2}$	$-\sqrt{10}$	2	0	$-\sqrt{3}$	0	$\sqrt{3}$	$-\sqrt{2}$	$\sqrt{3}$	0	-2	0	0	0	$\sqrt{2}$
$^3D_{3/2}$	$2\sqrt{15}$	0	0	$\sqrt{2}$	0	$\sqrt{2}$	0	$\sqrt{2}$	2	$-\sqrt{6}$	0	2	0	$-6\sqrt{3}$
$^3D_{3/2}(210\times21)$	0	$-12\sqrt{2}$	$-3\sqrt{15}$	$2\sqrt{6}$	$-4\sqrt{15}$	$2\sqrt{6}$	8	$\sqrt{6}$	$-6\sqrt{3}$	0	$-4\sqrt{30}$	$12\sqrt{3}$	$7\sqrt{10}$	0
$^3D_{3/2}(210\times20)$	0	$17\sqrt{3}$	$9\sqrt{10}$	-26	$-9\sqrt{10}$	-26	$6\sqrt{6}$	22	$4\sqrt{2}$	0	$3\sqrt{5}$	$-8\sqrt{2}$	0	0
$^3P_{3/2}$	0	$-\sqrt{3}$	$-\sqrt{10}$	6	$-\sqrt{10}$	-6	0	2	$2\sqrt{2}$	0	$\sqrt{5}$	0	0	0
$^1S_{3/2}$	0	0	0	$\sqrt{2}$	0	$\sqrt{2}$	0	$\sqrt{2}$	-1	0	0	-1	0	0
$^1D_{1/2}$	$2\sqrt{30}$	$\sqrt{3}$	0	-2	0	-2	$3\sqrt{6}$	-2	$-2\sqrt{2}$	$\sqrt{3}$	$-3\sqrt{5}$	$-2\sqrt{2}$	0	$3\sqrt{6}$
$^3P_{1/2}$	0	$-\sqrt{6}$	$-2\sqrt{5}$	$-3\sqrt{2}$	$-2\sqrt{5}$	$3\sqrt{2}$	0	$-\sqrt{2}$	-2	0	$\sqrt{10}$	0	0	0

TABLE 9. $M = \frac{1}{2}$ – Continued

	$\bar{2}1-\bar{2}$	$\bar{2}1-3$	$2\bar{1}-\bar{1}$	$\bar{2}\bar{1}-2$	$\bar{2}0-\bar{1}$	$\bar{2}0-2$	$\bar{2}\bar{0}-1$	$1\bar{1}-\bar{1}$	$1\bar{1}-2$	$10-\bar{1}$	$10-2$	$1\bar{0}-1$	$\bar{1}00$	$\bar{1}0-1$
$^2L_{17/2}$	$-12\sqrt{2}$	$-4\sqrt{3}$	0	$3\sqrt{2}$	$-6\sqrt{15}$	$-4\sqrt{6}$	$4\sqrt{15}$	$15\sqrt{2}$	$9\sqrt{5}$	$-5\sqrt{6}$	0	$10\sqrt{6}$	$-10\sqrt{2}$	$-5\sqrt{6}$
$^2L_{15/2}$	$-36\sqrt{\frac{1}{3}}$	$16\sqrt{\frac{2}{3}}$	0	$9\sqrt{\frac{3}{5}}$	$-27\sqrt{\frac{10}{11}}$	32	$18\sqrt{\frac{10}{11}}$	$45\sqrt{\frac{3}{11}}$	$-12\sqrt{\frac{30}{11}}$	40	0	-80	$-30\sqrt{\frac{3}{11}}$	40
$^2K_{15/2}$	$-20\sqrt{3}$	$-32\sqrt{2}$	0	$-19\sqrt{3}$	$9\sqrt{10}$	-64	$-22\sqrt{10}$	$25\sqrt{3}$	$12\sqrt{30}$	40	0	0	$10\sqrt{5}$	-40
$^4I_{15/2}$	$-6\sqrt{\frac{6}{11}}$	-10	$-3\sqrt{15}$	$-6\sqrt{\frac{6}{11}}$	$-6\sqrt{\frac{5}{11}}$	$-16\sqrt{\frac{2}{11}}$	$-6\sqrt{\frac{5}{11}}$	0	0	$-5\sqrt{\frac{2}{11}}$	$-6\sqrt{5}$	$-5\sqrt{2}$	0	$-5\sqrt{\frac{5}{11}}$
$^2K_{13/2}$	$-20\sqrt{6}$	56	0	$-19\sqrt{6}$	$18\sqrt{5}$	$56\sqrt{2}$	$-44\sqrt{5}$	$25\sqrt{6}$	$-21\sqrt{15}$	$-35\sqrt{2}$	0	0	$10\sqrt{6}$	$35\sqrt{2}$
$^4I_{13/2}$	$-9\sqrt{\frac{6}{11}}$	10	$-12\sqrt{15}$	$-9\sqrt{\frac{6}{11}}$	$-9\sqrt{\frac{5}{11}}$	$16\sqrt{2}$	$-9\sqrt{\frac{5}{11}}$	0	0	$5\sqrt{\frac{2}{11}}$	$21\sqrt{5}$	$5\sqrt{2}$	0	$5\sqrt{2}$
$^2I_{13/2}$	$-18\sqrt{3}$	$-28\sqrt{2}$	0	$6\sqrt{3}$	$6\sqrt{10}$	-14	$-3\sqrt{10}$	$-30\sqrt{3}$	0	35	0	-70	$30\sqrt{3}$	35
$^4I_{11/2}$	12	$5\sqrt{6}$	$-18\sqrt{10}$	12	$2\sqrt{30}$	$16\sqrt{3}$	$2\sqrt{30}$	0	0	$5\sqrt{3}$	$-9\sqrt{30}$	$5\sqrt{3}$	0	$5\sqrt{3}$
$^2I_{11/2}$	$-9\sqrt{2}$	$8\sqrt{3}$	0	$3\sqrt{2}$	$2\sqrt{15}$	$2\sqrt{6}$	$-\sqrt{15}$	$-15\sqrt{2}$	0	$-5\sqrt{6}$	0	$10\sqrt{6}$	$15\sqrt{2}$	$-5\sqrt{6}$
$^2H_{11/2}(210\times21)$	$4\sqrt{\frac{5}{11}}$	$-\sqrt{30}$	0	$-4\sqrt{\frac{5}{11}}$	$-3\sqrt{6}$	$-2\sqrt{15}$	$3\sqrt{6}$	$-5\sqrt{\frac{5}{11}}$	$-9\sqrt{2}$	$-2\sqrt{15}$	0	0	$-2\sqrt{5}$	$2\sqrt{15}$
$^2H_{11/2}(210\times11)$	$-5\sqrt{2}$	$-8\sqrt{3}$	0	$5\sqrt{2}$	0	$4\sqrt{6}$	$3\sqrt{15}$	$-5\sqrt{2}$	0	$-5\sqrt{6}$	0	0	$-5\sqrt{2}$	$5\sqrt{6}$
$^4G_{11/2}$	$-\sqrt{30}$	$3\sqrt{5}$	$-7\sqrt{3}$	$-\sqrt{30}$	-16	$-4\sqrt{10}$	-16	0	0	$-4\sqrt{10}$	-16	$-4\sqrt{10}$	0	$-4\sqrt{10}$
$^4I_{9/2}$	$-21\sqrt{10}$	$10\sqrt{15}$	84	$-21\sqrt{10}$	$-35\sqrt{3}$	$16\sqrt{30}$	$-35\sqrt{3}$	0	0	$5\sqrt{30}$	$-35\sqrt{3}$	$5\sqrt{30}$	0	$5\sqrt{30}$
$^2H_{9/2}(210\times21)$	$-4\sqrt{30}$	$-5\sqrt{5}$	0	$4\sqrt{30}$	18	$-5\sqrt{10}$	-18	$5\sqrt{30}$	$-15\sqrt{3}$	$-5\sqrt{10}$	0	0	$2\sqrt{30}$	$5\sqrt{10}$
$^2H_{9/2}(210\times11)$	$2\sqrt{15}$	$-4\sqrt{10}$	0	$-2\sqrt{15}$	0	$4\sqrt{5}$	$-9\sqrt{2}$	$2\sqrt{15}$	0	$-5\sqrt{5}$	0	0	$2\sqrt{15}$	$5\sqrt{5}$
$^4G_{9/2}$	$7\sqrt{15}$	$6\sqrt{10}$	$63\sqrt{6}$	$7\sqrt{15}$	$56\sqrt{2}$	$-16\sqrt{5}$	$56\sqrt{2}$	0	0	$-16\sqrt{5}$	$-120\sqrt{2}$	$-16\sqrt{5}$	0	$-16\sqrt{5}$
$^4G_{9/2}(210\times21)$	$-9\sqrt{30}$	$-5\sqrt{5}$	0	$-\sqrt{30}$	-22	$-5\sqrt{10}$	32	$-5\sqrt{30}$	$35\sqrt{3}$	$10\sqrt{10}$	0	$-20\sqrt{10}$	$12\sqrt{30}$	$10\sqrt{10}$
$^4G_{9/2}(210\times20)$	$-8\sqrt{15}$	$18\sqrt{10}$	0	$10\sqrt{15}$	$-16\sqrt{2}$	$-13\sqrt{5}$	$8\sqrt{2}$	$-6\sqrt{15}$	0	$5\sqrt{5}$	0	$-10\sqrt{5}$	$6\sqrt{15}$	$5\sqrt{5}$
$^4F_{9/2}$	$-\sqrt{15}$	$-2\sqrt{10}$	$-\sqrt{6}$	$-\sqrt{15}$	0	0	0	0	0	0	0	0	0	0
$^4G_{7/2}$	$\sqrt{30}$	$-6\sqrt{5}$	$-63\sqrt{3}$	$\sqrt{30}$	16	$8\sqrt{10}$	16	0	0	$8\sqrt{10}$	-96	$8\sqrt{10}$	0	$8\sqrt{10}$
$^4G_{7/2}(210\times21)$	$9\sqrt{15}$	$-2\sqrt{10}$	0	$\sqrt{15}$	$11\sqrt{2}$	$-4\sqrt{5}$	$-16\sqrt{2}$	$5\sqrt{15}$	$14\sqrt{6}$	$8\sqrt{5}$	0	$-16\sqrt{5}$	$-12\sqrt{15}$	$8\sqrt{5}$
$^4G_{7/2}(210\times20)$	$20\sqrt{\frac{3}{11}}$	$36\sqrt{2}$	0	$-25\sqrt{3}$	$8\sqrt{10}$	-26	$-4\sqrt{10}$	$15\sqrt{3}$	0	10	0	-20	$-15\sqrt{3}$	10
$^4F_{7/2}$	$\sqrt{6}$	-2	$\sqrt{15}$	$\sqrt{6}$	0	0	0	0	0	0	0	0	0	0
$^2F_{7/2}(210\times21)$	$-3\sqrt{3}$	$2\sqrt{2}$	0	$\sqrt{3}$	$3\sqrt{10}$	4	0	$\sqrt{3}$	$2\sqrt{30}$	-8	0	0	$-4\sqrt{3}$	8
$^2F_{7/2}(100\times10)$	0	0	0	$\sqrt{3}$	0	2	0	$\sqrt{3}$	0	2	0	0	$\sqrt{3}$	-2
$^4D_{7/2}$	-6	$-\sqrt{6}$	0	-6	$\sqrt{30}$	$-2\sqrt{3}$	$\sqrt{30}$	0	0	$5\sqrt{3}$	$\sqrt{30}$	$5\sqrt{3}$	0	$5\sqrt{3}$
$^4G_{5/2}$	$5\sqrt{3}$	$6\sqrt{2}$	$-7\sqrt{30}$	$5\sqrt{3}$	$8\sqrt{10}$	-16	$8\sqrt{10}$	0	0	-16	$8\sqrt{10}$	-16	0	-16
$^4F_{5/2}$	-1	$-2\sqrt{6}$	$3\sqrt{10}$	-1	0	0	0	0	0	0	0	0	0	0
$^2F_{5/2}(210\times21)$	$-3\sqrt{2}$	$-\sqrt{3}$	0	$\sqrt{2}$	$2\sqrt{15}$	$-\sqrt{6}$	0	$\sqrt{2}$	$-3\sqrt{5}$	$2\sqrt{6}$	0	0	$-4\sqrt{2}$	$-2\sqrt{6}$
$^2F_{5/2}(100\times10)$	0	0	0	2	0	$-\sqrt{3}$	0	2	0	$-\sqrt{3}$	0	0	2	$\sqrt{3}$
$^4D_{5/2}$	$-15\sqrt{2}$	$2\sqrt{3}$	0	$-15\sqrt{2}$	$5\sqrt{15}$	$2\sqrt{6}$	$5\sqrt{15}$	0	0	$-5\sqrt{6}$	$-9\sqrt{15}$	$-5\sqrt{6}$	0	$-5\sqrt{6}$
$^2D_{5/2}(210\times21)$	$4\sqrt{2}$	$9\sqrt{3}$	0	$-12\sqrt{2}$	$2\sqrt{15}$	$9\sqrt{6}$	$6\sqrt{15}$	$17\sqrt{2}$	$21\sqrt{5}$	$3\sqrt{6}$	0	$-6\sqrt{6}$	$18\sqrt{2}$	$3\sqrt{6}$
$^2D_{5/2}(210\times20)$	6	$-2\sqrt{6}$	0	-18	$-2\sqrt{30}$	$-4\sqrt{3}$	$\sqrt{30}$	-6	0	$\sqrt{3}$	0	$-2\sqrt{3}$	6	$\sqrt{3}$
$^4F_{3/2}$	$-\sqrt{2}$	$\sqrt{\frac{3}{11}}$	$\sqrt{5}$	$-\sqrt{2}$	0	0	0	0	0	0	0	0	0	0
$^4D_{3/2}$	0	$\sqrt{2}$	0	0	0	2	0	0	0	-5	$3\sqrt{10}$	-5	0	-5
$^2D_{3/2}(210\times21)$	4	$-3\sqrt{6}$	0	-12	$\sqrt{30}$	$-6\sqrt{3}$	$3\sqrt{30}$	17	$-7\sqrt{10}$	$-2\sqrt{3}$	0	$4\sqrt{3}$	18	$-2\sqrt{3}$
$^2D_{3/2}(210\times20)$	$3\sqrt{6}$	4	0	$-9\sqrt{6}$	$-6\sqrt{5}$	$4\sqrt{2}$	$3\sqrt{5}$	$-3\sqrt{6}$	0	$-\sqrt{2}$	0	$2\sqrt{2}$	$3\sqrt{6}$	$-\sqrt{2}$
$^2P_{3/2}$	$-\sqrt{6}$	$\sqrt{\frac{4}{11}}$	0	$\sqrt{6}$	0	$-2\sqrt{2}$	$-\sqrt{5}$	$-\sqrt{6}$	0	$-\sqrt{2}$	0	0	$-\sqrt{6}$	$\sqrt{2}$
$^4S_{3/2}$	0	$\sqrt{2}$	0	0	0	-1	0	0	0	1	0	1	0	1
$^4D_{1/2}$	$3\sqrt{\frac{6}{11}}$	-2	0	$3\sqrt{\frac{6}{11}}$	$-3\sqrt{5}$	$-2\sqrt{2}$	$-3\sqrt{\frac{5}{11}}$	0	0	$5\sqrt{2}$	$-3\sqrt{5}$	$5\sqrt{2}$	0	$5\sqrt{2}$
$^2P_{1/2}$	$-2\sqrt{3}$	$-2\sqrt{2}$	0	$2\sqrt{3}$	0	2	$-\sqrt{10}$	$-2\sqrt{3}$	0	1	0	0	$-2\sqrt{3}$	-1

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Zeros of First Derivatives of Bessel Functions of the First Kind, $J_n'(x)$, $21 \leq n \leq 51$, $0 \leq x \leq 100$

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A table of zeros of first derivatives of Bessel functions of the first kind, $J_n'(x)$, is presented for $21 \leq n \leq 51$, $0 \leq x \leq 100$. A brief discussion of table generation and accuracy is included.

1. Introduction

Zeros of

$$J_n'(x)=0,\ (n=0,\ 1,\ 2,\ 3,\ \ldots) \tag{1}$$

are frequently required for the solution of problems in mathematical physics and engineering. Typical problems of this kind occur in the theory of heat conduction [1],[1] hydrodynamics, finite Hankel transforms, Fourier-Bessel expansions, etc.

In conjunction with a heat transfer problem [2], the authors conducted a literature search in the spring of 1961 to find a table of zeros of $J_n'(x)$. No adequate tables were found at that time, and a table of such zeros for $0 \leq n \leq 51$, $0 \leq x \leq 100$ was generated. It was subsequently learned that the new Royal Society Tables [3], published in 1960 and available in 1961, contained zeros of $J_n'(x)$ for orders $0 \leq n \leq 20$. For this reason the present table is given for orders $21 \leq n \leq 51$ and $0 \leq x \leq 100$. A thorough summary of existing tables is given on page 411 of volume I [4].

2. Determination of the Positive Roots of $J_n'(x)$

As is well known, all the roots of $J_n'(x)=0$, $n>-1$, are real (see [5]). The determination of these roots for $0<x<100$ and $n=21, 22, 23, \ldots, 51$ was carried out on the IBM 1620 computer. The method employed was a modification of Newton's method. If x_k is an approximate root of $F(x)=0$, an improved value, x_{k+1}, is found by Newton's method as

$$x_{k+1}=x_k-\frac{F(x_k)}{F'(x_k)}.$$

The modified form of the method used in the computation of table 1 found x_{k+1} from

$$x_{k+1}=x_k-\frac{J_n'(x_k)}{\alpha},$$

where

$$\alpha=\frac{J_n'(x_0+0.005)-J_n'(x_0-0.005)}{0.01},$$

and x_0 is the first approximation of a root of equation (1). $J_n'(x)$ as used in this formula was found by a truncation of the series

$$J_n'(x)=\sum_{m=0}^{\infty}(-1)^m\frac{(2m+n)\left(\frac{x}{2}\right)^{2m+n-1}}{2(m!)(m+n)!}.$$

The series was truncated to produce $J_n'(x)$ accurate at least to the order of 10^{-15}.

The first approximation, x_0, to a root of $J_n'(x)$ was obtained from the Harvard tables [6]. The iteration discussed above was carried out until the change in the approximate value of a root between successive iterations had become less than 5×10^{-9}, or 10 iterations had been performed, whichever came first.

Machine running time was approximately one minute per root for small s increasing to approximately eight minutes per root for large s. Advantage was taken of the variable word length feature of the IBM 1620 computer, and 50 significant figures were carried in the computer.

The use of asymptotic (McMahon) series to compute the zeros of $J_n'(x)=0$ was also tried. When using the first four terms of the appropriate series, the accuracy obtained was not adequate for present purposes. For instance, when $n=0$, agreement to four decimal places requires that $s \geq 3$, but when $n=15$, agreement to four decimal places requires that $s \geq 18$. This method of computation was found to be awkward and was subsequently abandoned.

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[1] Figures in brackets indicate the literature references at the end of this paper.

TABLE 1. Zeros $j'_{n,s}$ of $J'_n(x)$

s \ n	21	22	23	24	25	26	27	28
1	23.25482051	24.28938554	25.32292421	26.35550968	27.38720713	28.41807483	29.44816521	30.47752554
2	28.81559026	29.91614781	31.01399843	32.10931996	33.20227214	34.29299903	35.38163108	36.46828685
3	33.11916251	34.26076830	35.39878197	36.53342770	37.66490790	38.79340601	39.91908887	41.04210875
4	37.05164392	38.22489744	39.39397753	40.55913306	41.72058970	42.87855284	44.03320998	45.18473289
5	40.78864004	41.98788278	43.18254757	44.37289611	45.55916665	46.74157681	47.92032600	49.09559743
6	44.40300395	45.62431208	46.84075435	48.05259808	49.26008750	50.46344645	51.66288070	52.85857983
7	47.93297704	49.17342240	50.40879603	51.63936621	52.86537891	54.08706022	55.30461857	56.51824657
8	51.40136592	52.65866144	53.91073936	55.15786592	56.40028590	57.63822500	58.87189183	60.10147963
9	54.82293308	56.09523556	57.36221966	58.62414749	59.88126077	61.13378297	62.38192112	63.62586750
10	58.20780036	59.49358979	60.77399447	62.04927114	63.31965714	64.58537241	65.84662124	67.10359375
11	61.56321210	62.86121252	64.15378858	65.44119072	66.72365106	68.00138519	69.27459380	70.54346404
12	64.89452838	66.20365410	67.50733617	68.80581881	70.09932836	71.38807549	72.6722567l	73.95205564
13	68.20582532	69.52513956	70.83900854	72.14766879	73.45134051	74.75022911	76.04452654	77.33441251
14	71.50026659	72.82895627	74.15221157	75.47026176	76.78332077	78.09158850	79.39525213	80.69448717
15	74.78035780	76.11770942	77.44964789	78.77639527	80.09815911	81.41513374	82.72750140	84.03543330
16	78.04811263	79.39349598	80.73349570	82.06832667	83.39819010	84.72327469	86.04375773	87.35980602
17	81.30517182	82.65802681	84.00553424	85.34790193	86.68532484	88.01798612	89.34605807	90.66970305
18	84.55288816	85.91271419	87.26723391	88.61664829	89.96114613	91.30090509	92.63609256	93.9668665
19	87.79238862	89.15873605	90.51982252	91.87584232	93.22697825	94.5734026	95.9152779	97.252758
20	91.02462083	92.39708397	93.76433469	95.12656080	96.4839394	97.83664	99.18481	
21	94.2503882	95.6285994	97.0016496	98.369720	99.73298			
22	97.470377	98.85400						

s \ n	29	30	31	32	33	34	35	36
1	31.50619868	32.53422356	33.56163572	34.58846766	35.61474922	36.64050785	37.66576885	38.69055565
2	37.55307443	38.63609270	39.71743238	40.79717690	41.87540324	42.95218254	44.02758073	45.10165905
3	42.16260504	43.28070570	44.39652854	45.51018228	46.62176750	47.73137747	48.83909883	49.94501228
4	46.33327932	47.47899456	48.62201276	49.76245810	50.90044573	52.03608272	53.16946879	54.30069698
5	50.26755989	51.43636931	52.60217002	53.76509596	54.92527166	56.08281316	57.23782876	58.39041974
6	54.05071903	55.23946052	56.42495487	57.60734209	58.78675269	59.96330850	61.13712347	62.30830435
7	57.72812261	58.93441230	60.13726972	61.33683844	62.53325256	63.72663752	64.91711085	66.10478288
8	61.32716786	62.54912345	63.76750203	64.98244897	66.19410026	67.40258336	68.60801790	69.81051636
9	64.86580101	66.10188848	67.33428573	68.56313857	69.78858366	71.01074932	72.22975619	73.44571783
10	68.35646723	69.60540730	70.85056897	72.09209754	73.33012943	74.56479294	75.79620886	77.02449111
11	71.80817077	73.06887767	74.32573818	75.57889638	76.82848778	78.07463998	79.31747334	80.55710149
12	75.22764416	76.49918344	77.76682482	79.03071064	80.29097497	81.54774425	82.80113789	84.05126881
13	78.62005552	79.90161382	81.17923628	82.45306311	83.72322656	84.98985154	86.25305616	87.51295225
14	81.98945851	83.28032121	84.56722136	85.85029678	87.12967762	88.40548697	89.67784138	90.94685132
15	85.33909047	86.63862466	87.93417897	89.22588862	90.51388147	91.79827859	93.07919474	94.35673887
16	88.67157668	89.97921796	91.28286992	92.58266499	93.87872862	95.17117970	96.46013108	97.7456900
17	91.98907425	93.30431637	94.61556629	95.92295364	97.2266013	98.526626	99.823138	
18	95.2933761	96.6157625	97.9341594	99.248694				
19	98.585988	99.915104						

s \ n	37	38	39	40	41	42	43	44
1	39.71488993	40.73879184	41.76228012	42.78537226	43.80808458	44.83043235	45.85242988	46.8740906l
2	46.17447445	47.24608006	48.31652548	49.38585712	50.45411848	51.52135039	52.58759122	53.65287710
3	51.04919307	52.15171154	53.25263353	54.35202077	55.44993122	56.54641941	57.64153668	58.73533146
4	55.42985428	56.55702215	57.68227699	58.80569058	59.92733042	61.04726012	62.16553970	63.28222582
5	59.54068094	60.68870138	61.83456466	66.96860305	64.12012999	65.25997613	66.39795402	67.53412615
6	63.47695133	64.64315855	65.80701462	70.82944334	68.12800265	72.00454949	70.44052920	71.59379334
7	67.28975730	68.47213170	69.65199810	74.59317900	72.00454949	75.78247092	74.34805104	75.51658976
8	71.01018463	72.20712253	73.40142430	78.28116712	75.78247092	79.48339598	78.15398172	79.33634823
9	74.65874137	75.86892789	77.07637299	81.90834900	79.48339598	83.12246507	81.88047903	83.07548378
10	78.24974724	79.47207895	80.69158247	85.48562731	83.12246507	86.71073237	85.54307039	86.74971206
11	81.79363188	83.02716621	84.2578008l	89.02122157	86.71073237	90.25653626	89.15310461	90.37052520
12	85.29824389	86.54216443	87.78312656	92.52149207	90.25653626	93.76633247	92.71915116	93.94660509
13	88.76964580	90.02323738	91.27382253	95.99146646	93.76633247	97.24522677	96.24785140	97.48468340
14	92.21262163	93.47525189	94.73483679	99.43519030	97.24522677		99.74446310	
15	95.63101442	96.90211981	98.17014868					
16	99.0279584							

s \ n	45	46	47	48	49	50	51	
1	47.89542716	48.91645142	49.93717458	50.95760723	51.97775935	52.99764039	54.01725930	
2	54.71724208	55.78071831	56.84333616	57.90512436	58.96611015	60.02631933	61.08577641	
3	59.82784946	60.91913392	62.00922575	63.09816371	64.18598460	65.27272333	66.35841310	
4	64.39737208	65.51102923	66.62324535	67.73406610	68.84353482	69.95169273	71.05857906	
5	68.66855175	69.80128692	70.93238495	72.06189644	73.18986951	74.31634998	75.44138148	
6	72.74514357	73.89463994	75.04233952	76.18829659	77.33256282	78.47518745	79.61621745	
7	76.68307658	77.84757445	79.01014324	80.17083993	81.32971886	82.48683185	83.64222836	
8	80.51654782	81.69464548	82.87070314	84.04477980	85.21693177	86.38721278	87.55567420	
9	84.26822496	85.45876907	86.64717947	87.83351663	89.01783829	90.20019962	91.38065337	
10	87.95400864	89.15602759	90.35583329	91.55348723	92.74904815	93.94257225	95.13411332	
11	91.58553174	92.79819229	94.00857188	95.21673264	96.42273401	97.62663288	98.82848375	
12	95.17158860	96.39416407	97.61440284	98.83236546				
13	98.71899365	99.9508506						

3. Tabulation Scheme: Accuracy Evaluation

Table 1 presents the computed roots. All entries were tested for accuracy to the number of places shown by observing the effect of perturbing the last digit of each entry. The digit which resulted in the smallest absolute value of the function was then selected. Thus, the only possibility of error when 50 decimal places are carried, is an accumulated round-off error, a computer malfunction, or a print-out error. Accumulated round-off errors can occur at larger x. Typographical errors have been minimized by several table checks. We note that the number of decimal places decreases for n small and s large. This is due to poor convergence of the successive iterations of the modified Newton's method used.

Where possible, the roots in our entire computation $0 \leq n \leq 51$, $0 \leq \times \leq 100$ were compared with those given in [7 and 8]. Fourteen of the larger roots of [7] do not agree with the corresponding results of our computation. The largest discrepancy occurs in the case of the seventh root of $J_3'(x) = 0$ which, in [7], is given as 24.1469, whereas the value by our method rounded off to four decimal places, is 24.1449. Six roots of [8] do not agree with our values. The largest discrepancy occurs in the third root $J_2'(x) = 0$, which, in [8] is given as 9.965, whereas the value by our computation, rounded off to three decimal places, is 9.969. To test for the chance of error in machine output, the 14 cases with discrepancies in [7] and the 6 cases of [8] were separately re-run by summing the series of $J_n'(x)$ with the values of our computation rounded off to four places and three places respectively, and then summing with the corresponding values of [7 and 8]. In every case the rounded-off values by the method of section 2 gave the better approximation to the root. These discrepancies are also discussed in [4 and 9].

Another check on the present computation was afforded by the application of the Lagrange interpolating polynomial used in conjunction with the Harvard Tables [6]. This method requires obtaining the Lagrange interpolating polynomial for $J_n'(x)$ through k points in the vicinity of the root. For each of the k values of x we obtain the value of $J_n'(x)$ from the formula $2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$, where $J_{n-1}(x)$ and $J_{n+1}(x)$ are obtained from [6]. The root of the interpolating polynomial was then found by a modification of Newton's method. When this method was used with $k = 4$ in a limited number of cases, ($n = 1$, $1 \leq s \leq 16$, and $n = 15$, $s = 1$), the roots so obtained coincided to six decimal places with those obtained by the computation of section 2. The Lagrange interpolation method was originally intended for the computation of table 1, but was abandoned because of the excessive labor required for data input to the computer from the Harvard Tables.

Finally, the values for $0 \leq n \leq 20$ were compared at all common values with the Royal Society Tables [3]. It was found that eight decimals agreed for all n and $x < 78$. For some n and $x > 78$ a last decimal round-off discrepancy is observed. It is concluded that the tables for $21 \leq n \leq 51$ are correct as presented for x smaller than about 80, and may err at the last decimal place shown for other entries.

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Publications of the National Bureau of Standards*

Selected Abstracts

On the fourth order Hamiltonian of an asymmetric rotor molecule of orthorhombic symmetry, Wm. B. Olson and H. C. Allen, Jr., *J. Research NBS* **67A** (*Phys. & Chem.*) *No. 4, 357–362 (July–Aug. 1963).*
The fourth order Hamiltonian of an asymmetric rotor molecule of orthorhombic symmetry given recently has been considerably reduced in complexity through the use of equations derived from the basic relationship among the angular momentum operators. The reduced Hamiltonian obtained provides a most convenient starting point for the calculation of rotational energy levels from a solution of the complete secular equation, for a perturbation theory solution to the problem of centrifugal distortion, and for the deduction of sum rules among the energy levels.

Asymptotic behavior of the current on an infinite cylindrical antenna, K. S. Kunz, *J. Research* **67D** (*Radio Prop.*) *No. 4, 417 (July–Aug. 1963).*
An asymptotic expression is obtained for the current distribution on the outside surface of an infinitely long, perfectly conducting, hollow cylindrical antenna that is fed by an infinitesimally narrow circumferential gap. This asymptotic expression involves two series. The first series is expressed in reciprocal powers of $\log(2|z|/j\Gamma^2 ka^2)$, where $|z|$ is the distance from the gap, $\log\Gamma$ is Euler's constant, k is the propagation constant, and a is the radius of the antenna. The second series is a similar series multiplied by $1(k|z|)$. The first series is dominant and its first five terms yield values for the magnitude and phase of the current that for even moderately thick antennas (circumferences as large as $\lambda/3$) are accurate to within about one percent in as close as $\lambda/3$ of the gap. This is shown by a comparison of the values of the current obtained from these terms with the numerically computed values of Duncan (1962). Asymptotic expressions for the current found in the literature resemble the first term of this dominant series and are accurate only at relatively large distances from the gap—except for very thin antennas.

A dipole approximation of the backscattering from a conductor in a semi-infinite dissipative medium, M. B. Kraichman, *J. Research NBS* **67D** (*Radio Prop.*) *No. 4, 433 (July–Aug. 1963).*
The backscattering of a uniform plane wave by a conductor in a semi-infinite dissipative medium is discussed. The conductor is assumed to act as both an electric and a magnetic dipole with moments which are obtained from the electric magnetic polarizabilities of the conductor, respectively. Using these induced moments, expressions are derived for the backscattered electric field at a point on the surface of the dissipative halfspace directly above the dipoles. Both harmonic and transient excitation are considered.

A survey of some mathematical models in the theory of reliability, G. H. Weiss, *Book, Statistical Theory of Reliability, Ed. M. Zelen, pp. 3–54 (Univ. of Wisconsin Press, Madison, Wis., 1963).*
This paper contains a survey of models for systems which are subject to various types of failures. Topics which are discussed are, topological aspects of reliability, time-dependent reliability, and the analysis of maintenance and inspection policies.

Hartree-Fock approximation of CH_4 and NH_4^+, M. Krauss, *J. Chem. Phys.* **38,** *No. 2, 564–565 (Jan. 15, 1963).*
Molecular energies, which approach the Hartree-Fock limit, have been computed for the CH_4 and NH_4^+ molecules at their equilibrium distances. These results verify the suitability of exponential quadratic functions as basis functions in molecular calculations.

Some stochastic processes in polymer systems, J. Mazur, *J. Chem. Phys.* **38,** *No. 1, 193–201 (Jan. 1, 1963).*
Certain distributional problems involving polymer configurations can be treated as special classes of stochastic processes known as regenerative processes. These processes have the property that the interval (of time or of length) between two consecutive events is a random variable. The method of regeneration point is applied to the problem of random crystallization of polymers and to the problem of force-length relationships in a one-dimensional simulation of a polymer network. By assuming that the events of randomly placing crystalline units on a polymer chain are statistically independent, and that the probability of the first event occurring in a given interval is a simple step function, the direct application of the method of regeneration point leads to a well-known equation from the theory of molecular distribution in one-dimensional hard-core fluids. In the second problem it is assumed that a single polymer chain consists of mesh points connected by flexible chains. A restriction is imposed that these mesh points cannot pass through each other. The molecular distribution functions for these mesh points are derived with the help of the regeneration-point process. By applying this method, the relationships of the network extension to the fixed force are derived. It is also found that the affine transformation rule for the force-biased distribution of chain lengths holds strictly only if the array represents a Poisson distribution of mesh points.

On the three-particle scattering operator in classical gases, J. Weinstock, *Physics Letters* **3,** *No. 6, 260–262 (Feb. 1, 1963).*
The three-particle scattering operator is the central quantity which must be calculated in order to determine the density dependence of transport coefficients in gases. A perturbation method is presented, for such a calculation, which is based upon the "binary collision expansion" of the "Greens function" form of the three-particle scattering operator. In addition, it is pointed out that the n-particle scattering operators of non-equilibrium statistical mechanics are actually the asymptotic forms of more general time dependent n-particle operators, and that a complete description of non-equilibrium phenomena may be obtained from a knowledge of the time dependence of these more general operators.

Roger Joseph Boscovich and the combination of observations, C. Eisenhart, *Actes Symp. Intern. R. J. Boskovic 1961, pp. 19–25 (1962).*
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Interactions matrix element in a shell model, U. Fano, F. Prats, and Z. Goldsmith, *Phys. Rev.* **129,** *No. 9, 2643–2652 (Mar. 16, 1963).*
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From basic considerations, a volume in signal space has been derived, applicable to any analyzer. It is a product of resolution limits in energy, frequency, and time, and depends functionally only upon the signal-to-noise ratio. The derivation demonstrates the rise of statistical features as $S/N \to 0$.

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On the fourth order Hamiltonian of an asymmetric rotor molecule of orthorhombic symmetry. Wm. B. Olson and H. C. Allen, Jr. (See above abstract.)

Measurement of the thickness and refractive index of very thin films and the optical properties of surfaces by ellipsometry. F. L. McCrackin, E. Passaglia, R. R. Stromberg, and H. L. Steinberg.

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Optimum reception pattern of the Beverage wave antenna at very low frequencies. E. W. Seeley.

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Asymptotic behavior of the current on an infinite cylindrical antenna. K. S. Kunz. (See above abstract.)

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